Numerical Modelling in Porous Media using the Virtual Element Method

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International Day of Women and Girls in Science

February 11, 2021

Outline

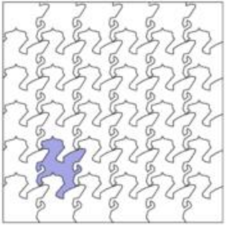
1. General Framework
2. Virtual Element Method: model problem
3. Virtual Element Method: applications in Porous Media
4. Conclusions

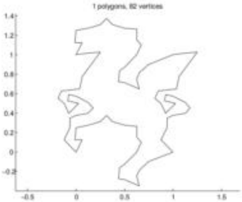
General Framework

- **Partial differential equations** (PDEs) are ubiquitous in mathematically-oriented scientific fields such as **physics** and **engineering**.
- **Analytical solutions** of PDEs are **NOT** always **available** $\Rightarrow$ solve the problem with **numerical methods**.
- A plethora of methods exist:
  - Finite Difference Method
  - Finite Element Method
  - Finite Volume Method
  - Spectral Method
  - Meshfree Method
  - ...
  - **Virtual Element Method (VEM)**

Virtual Element Method (VEM)

- The **Virtual Element Method** (VEM) is a **very recent** numerical method for solving PDEs.
  - Seminal paper:
    - L. Beirão da Veiga, F. Brezzi, A. Cangiani, G. Manzini, L.D. Marini, A.Russo: **Basic principles of Virtual Element Methods**, Mathematical Models and Methods in Applied Sciences, **2013**.
- The VEM **generalizes** the Finite Element Method (FEM)
  - **FEM** faces **considerable difficulties** on general polygonal meshes;
  - **VEM** allows to solve PDEs on very **general polygonal** meshes.





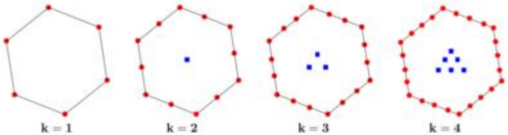
VEM: model problem

- **Continuous problem**:
$$-\Delta u = f \text{ in } \Omega \subset \mathbb{R}^2, \quad u = 0 \text{ on } \partial\Omega.$$
- **Variational formulation**:
$$\text{find } u \in H_0^1(\Omega) \ \text{ s.t. } \ a(u,v) = (f,v), \ \ \forall v \in H_0^1(\Omega),$$
$a(u,v) = (\nabla u, \nabla v)$ bilinear form and $(\cdot,\cdot)$ = scalar product in L^2.
- **VEM** fundamental **ingredients**:
  - A **decomposition** (mesh) $\mathcal{T}_\delta$ of Ω into **polygons** E.
  - A finite dimensional functional space $V_\delta^k \subset H_0^1(\Omega)$ (**global virtual element space**).
  - A bilinear form $a_\delta : V_\delta^k \times V_\delta^k \to \mathbb{R}$ that can be split over the polygons, i.e., $a_\delta(u_\delta, v_\delta) = \sum_{E \in \mathcal{T}_\delta} a_\delta^E(u_\delta, v_\delta)$, $u_\delta, v_\delta \in V_\delta^k$.
    The local bilinear form a_δ^E must be **computable** and must satisfy
    - **polynomial consistency**
    - **stability**
  - Construction of the right-hand side.

VEM: model problem

Virtual Element Space

- The global space is constructed from the local spaces.
- On each polygon E, define a **local virtual element space** $\mathcal{V}_\delta^{k,E}$:
  - polynomials of degree k + additional functions solution of a suitable PDE inside E.
- Take the following **degrees of freedom** in $\mathcal{V}_\delta^{k,E}$. Let $v_\delta \in \mathcal{V}_\delta^{k,E}$
  1. values of v_δ at the vertices of E;
  2. for $k > 1$, values of v_δ at the $k-1$ internal points of the Gauss-Lobatto quadrature rule on each edge e;
  3. for $k > 1$, the momentum $\frac{1}{|E|}\int_E v_\delta \; m \, \mathrm{d}\mathbf{x}, \quad m \in \mathcal{M}_{k-2}(E)$ set of scaled monomials of degree $\leq k-2$.



VEM: model problem

- Introduce the following orthogonal projection operators:
  - $H^1(E)$-**orthogonal projection operator** $\Pi^{\nabla}_{k,E} : H^1(E) \to \mathbb{P}_k(E)$.
  - $L^2(E)$-**orthogonal projection operator** $\Pi^{0}_{k-1,E} : L^2(E) \to \mathbb{P}_{k-1}(E)$.
  - Computable using the degrees of freedom.
- A choice of the local bilinear form a^E_δ that satisfies consistency and stability

$$a^E_\delta(u_\delta, v_\delta) := (\nabla\Pi^{\nabla}_{k,E} u_\delta, \nabla\Pi^{\nabla}_{k,E} v_\delta) + S^E((I - \Pi^{\nabla}_{k,E})u_\delta, (I - \Pi^{\nabla}_{k,E})v_\delta),$$

  for the stabilization form S^E different choices exist.
- A choice for the right-hand side

$$(f, \Pi^0_{k-1} v_\delta).$$

- Finally, the **VEM discrete variational formulation** reads

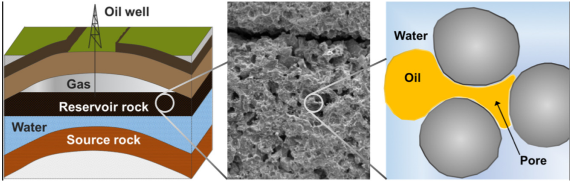
$$\text{find } u_\delta \in V^k_\delta \ s.t. \ \sum_{E\in\mathcal{T}_\delta} a^E_\delta(u_\delta, v_\delta) = (f, \Pi^0_{k-1} v_\delta), \ \ \forall v_\delta \in V^k_\delta.$$

Two-phase flow in porous media

- Complex and realistic geological **flow models** in **porous media**
  - **large scale problems**, domains displaying **complex geometries**.
- Idea: investigate the potentialities of **VEM** in the contest of **two-phase flow** of **immiscible fluids** in porous media.
  - Applications: petroleum and chemical engineering, hydrology, nuclear waste disposal safety.

Concepts:

- **Porous medium** = porous matrix + void space.
- **Two-phase flow**: void space filled by two immiscible fluids
  - wetting phase (w);
  - non-wetting phase (n).



Two-phase flow equations

For each phase ($\alpha = w, n$) find S_α, (saturation), u_α (Darcy's velocity) and p_α (pressure) in the space-time domain $Q_T = \Omega \times \mathcal{I}_T$ s.t.

$$\begin{cases} \frac{\partial(\Phi \rho_\alpha S_\alpha)}{\partial t} + \nabla \cdot (\rho_\alpha \mathbf{u}_\alpha) = \rho_\alpha q_\alpha \\ \mathbf{u}_\alpha = -\frac{k_{r_\alpha}}{\mu_\alpha} \mathbf{K}(\nabla p_\alpha - \rho_\alpha \mathbf{g}) \\ S_w + S_n = 1 \\ p_n - p_w = p_c \\ \; + \textit{boundary and initial conditions.} \end{cases} \tag{1}$$

$p_c(S_\alpha)$ and $k_{r_\alpha}(S_\alpha) \Rightarrow$ *Brooks-Corey empirical model.*

Equations (1) are:

- **time dependent**;
- **non-linear**;
- **coupled**.

Rewrite (1) using the **pressure-saturation formulation** $(p_n - S_w)$

- Goal: find S_w and p_n

$$\begin{cases} -\nabla\cdot\left\{\mathbf{K}\lambda\nabla p_n - \mathbf{K}\lambda_w\frac{dp_c}{dS_w}\nabla S_w + \mathbf{K}(\lambda_w\rho_w + \lambda_n\rho_n)\mathbf{g}\right)\right\} = q, \\ \Phi\frac{\partial S_w}{\partial t} + \nabla\cdot\left\{\mathbf{K}\lambda_w\frac{dp_c}{dS}\nabla S_w - \mathbf{K}\lambda_w\nabla p_n + \mathbf{K}\lambda_w\rho_w\mathbf{g}\right\} = q_w, \\ + \textit{ boundary and initial conditions.} \end{cases} \tag{2}$$

- Pressure equation $\Rightarrow$ **elliptic** w.r.t. p_n;
- Saturation equation $\Rightarrow$ **non-linear hyperbolic** $(p_c = 0)$ or **parabolic** $(p_c \neq 0)$ w.r.t. S_w.

Pressure-Saturation Formulation

Equations (2) are **non-linearly coupled**

- Traditional approaches:
  - Fully Implicit Method (FIM) or operator splitting techniques (IMPES, IMPIS) + classical space discretization scheme (Finite Elements, Finite Volumes, Discontinuous Galerkin methods).
- Our approach:
  - **Iterative IM**plicit-**P**ressure-**I**mplicit-**S**aturation (IMPIS) method + **V**irtual **E**lement **M**ethod (VEM)
    1. The time discretization with **Crank-Nicolson** scheme gives rise to a fully implicit and coupled system;
    2. Linearize the saturation equation with respect to the saturation using **Newton-Raphson** method and adopt an **iterative IMPIS formulation** to split the pressure and the saturation equations and solve them iteratively.
    3. Discretize in space using the **Virtual Element Method**.

Analysis of the **convergence** of the method in case the **analytical solution** is known.

- **Space** and **time domain**

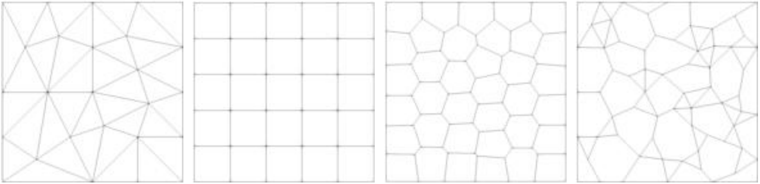
$$\Omega = (0,1) \times (0,1)\ [m^2], \quad \mathcal{I}_T = [0,1]\ [s].$$

- **Analytical solutions**

$$p_{n_{ex}}(x,y,t) = 10^5 \cdot t\ x(1-x)y(1-y) \quad [Pa],$$
$$S_{w_{ex}}(x,y,t) = \frac{1}{2} + t\ x(1-x)y(1-y) \quad [-].$$

- **Physical data** for the porous medium and the fluids: porosity (Φ), absolute permeability ($\mathbf{K}$), residual saturations (S_{wr}, S_{nr}), viscosities (μ_w, μ_n), Brooks-Corey parameters (μ, p_d).

- Four meshes made up of different types of polygonal tessellations



(a) Triangle (b) Square (c) Polygon (d) Agglomerated

Figure 1: Meshes.

Numerical Experiments - analytical solution

- Order of convergence in H^1 norm

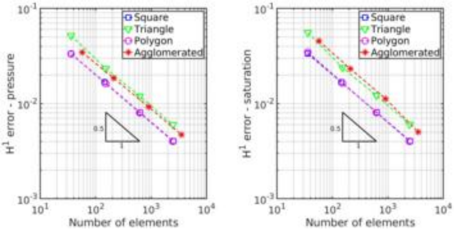


Figure 2: VEM order $k = 1$.

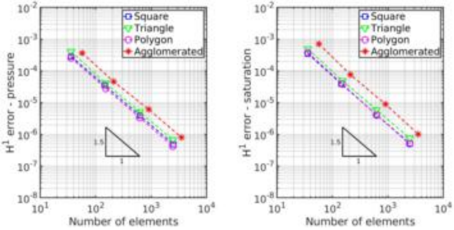


Figure 3: VEM order $k = 3$.

Benchmark problem

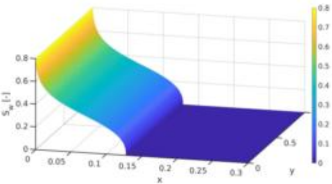
A **benchmark problem** for two-phase flow having quasi-analytical solution (**parabolic case**).

- **McWhorter and Sunada** ($p_c \neq 0$) - **Bidirectional flow**

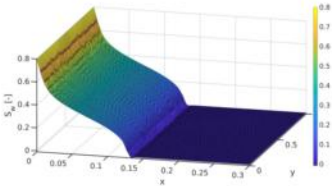
$$\begin{cases} u(x,t) = 0, \\ \Phi \dfrac{\partial S_w}{\partial t} + \dfrac{\partial}{\partial x}\left(K f_w \lambda_n \dfrac{dp_c}{dS_w} \dfrac{\partial S_w}{\partial x} \right) = 0. \end{cases}$$

- It involves the flow of two immiscible and incompressible fluids (water and oil) through a one-dimensional horizontal porous medium representing a reservoir.
- Originally mono-dimensional $\Rightarrow$ to verify the code on a bi-dimensional domain, solve a 2D problem assuming constant solution on the y direction.

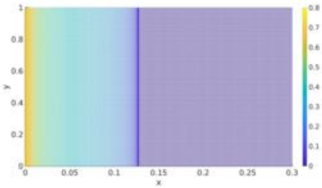
- Polygonal mesh



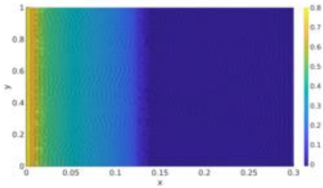
(a) S_w: semi-analytical solution.



(b) S_w: numerical solution.



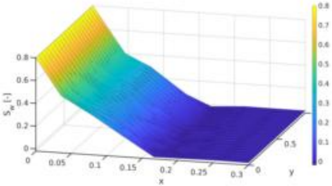
(c) S_w: semi-analytical solution.



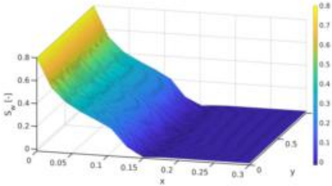
(d) S_w: numerical solution.

Benchmark problem (Polygonal mesh refinement)

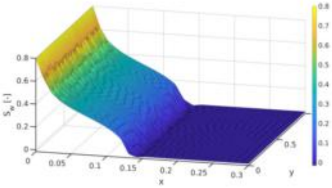
- Qualitative converge of the numerical saturation to the semi-analytical solution through mesh **refinement**.



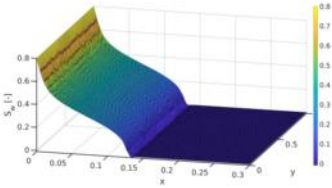
(e) $\mathcal{N}_\delta = 154,\ \Delta t = 10\ s.$



(f) $\mathcal{N}_\delta = 619,\ \Delta t = 5\ s.$



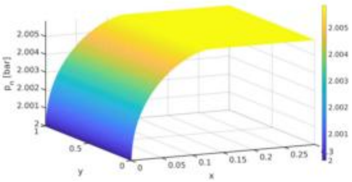
(g) $\mathcal{N}_\delta = 2463,\ \Delta t = 2.5\ s.$



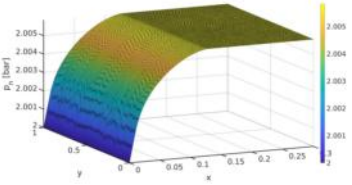
(h) $\mathcal{N}_\delta = 9911,\ \Delta t = 1.25\ s.$

Benchmark problem (Pressure p_n, $k = 1$)

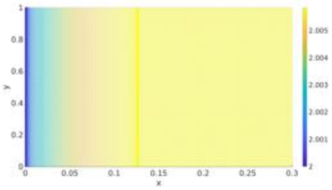
- Polygonal mesh



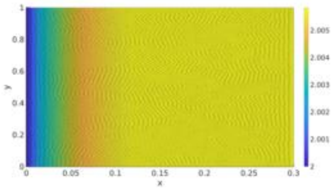
(i) p_n: semi-analytical solution.



(j) p_n: numerical solution.



(k) p_n: semi-analytical solution.



(l) p_n: numerical solution.

Conclusions

- VEM is a new method and a lot of work is still needed to assess **pros** and **cons**.
- Many VEM key features make VEM a **competitive alternative** to other numerical methods for PDEs.
- VEM is a numerical tool with **potentialities** in solving complex and realistic geological flow models.
- A lot more is still to be explored!

Thank you for your attention!

"Don't let anyone rob you of your imagination, your creativity, or your curiosity. It's your place in the world; it's your life. Go on and do all you can with it, and make it the life you want to live."
Mae Jemison - first African American woman astronaut.