An overview of separable functors and their applications

PhD in Pure and Applied Mathematics - XXXVI cycle
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Introduction

1. Categories and functors
2. Separable functors and applications
3. How can we extend this notion?

Categories and Functors

- A category $\mathcal{C}$ is the datum of a class of objects $\mathrm{Ob}(\mathcal{C})$ and a class of morphisms between them such that for any ordered pair $(X, Y) \in \mathrm{Ob}(\mathcal{C})$ there is a set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ of morphisms of X in Y, satisfying some requirements.
- Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A functor $F : \mathcal{C} \to \mathcal{D}$ consists of:
  - a map
  $$Ob(\mathcal{C}) \to Ob(\mathcal{D}), \quad X \mapsto F(X)$$
  - for every ordered pair $(X, Y) \in \mathrm{Ob}(\mathcal{C})$, there is a map
  $$\mathcal{F}_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y)), \quad f \mapsto F(f)$$

  such that

  (i) $F(f \circ g) = F(f) \circ F(g)$, for every $f, g \in \mathcal{C}$ composable;
  (ii) $F(\mathrm{Id}_X) = \mathrm{Id}_{F(X)}$, for every object $X \in \mathcal{C}$.

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Examples

- $\mathrm{Id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$, $X \mapsto X$, $f \mapsto f$
- Forgetful functors: e.g. $U : \mathrm{Grp} \to \mathrm{Set}$, $(G, \cdot, 1) \mapsto G$, $f \mapsto f$
- Let $\varphi : R \to S$ be a ring homomorphism: $\varphi^* := (-) \otimes_R S : \mathcal{M}_R \to \mathcal{M}_S$, $M \mapsto M \otimes_R S$, $f \mapsto f \otimes_R \mathrm{Id}_S$, is the induction functor.

Adjunctions

Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. A natural transformation $\eta : F \to G$ is a class of morphisms $(\eta_X)_{X \in \mathcal{C}}$ in $\mathcal{D}$ such that, for every $f : X \to Y$ in $\mathcal{C}$,

$$\begin{tikzcd} F(X) \arrow[r, "\eta_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\eta_Y"'] & G(Y) \end{tikzcd}$$

commutes. A pair of functors $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ is an adjunction ($F \dashv G$) if there are natural transformations $\eta : \mathrm{Id}_{\mathcal{C}} \to GF$ (the *unit*) and $\epsilon : FG \to \mathrm{Id}_{\mathcal{D}}$ (the *counit*) such that $G\epsilon \circ \eta G = Id_G$ and $\epsilon F \circ F\eta = \mathrm{Id}_F$.

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Example

$\begin{tikzcd} \mathcal{M}_R \arrow[r, shift left, "\varphi^* = (-)\otimes_R S"] \arrow[r, phantom, "\perp"] & \mathcal{M}_S \arrow[l, shift left, "\varphi_*"] \end{tikzcd}$, φ_* is the restriction of scalars functor, the components of the unit and the counit are given by

$$\eta_M : M \to M \otimes_R S, \quad \eta_M(m) = m \otimes_R 1_S,$$

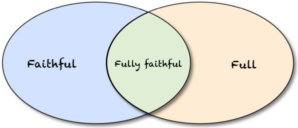
$$\epsilon_N : N \otimes_R S \to N, \quad \epsilon_N(n \otimes_R s) = ns.$$

Special functors

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor and consider the associated natural transformation

$$\mathcal{F} : \mathrm{Hom}_{\mathcal{C}}(-,-) \to \mathrm{Hom}_{\mathcal{D}}(F-,F-), \quad \mathcal{F}_{X,Y}(f) = F(f)$$

for any $f : X \to Y$ in $\mathcal{C}$. Then F is



faithful if $\mathcal{F}_{X,Y}$ is injective for every X, Y in $\mathcal{C}$, i.e. $F(f) = F(g) \Longrightarrow f = g$, for every $f, g : X \to Y$ in $\mathcal{C}$

- Forgetful functors, e.g.
  $U : \mathrm{Grp} \to \mathrm{Set}$
  $\varphi_* : \mathcal{M}_S \to \mathcal{M}_R$

full if $\mathcal{F}_{X,Y}$ is surjective for every $X, Y \in \mathcal{C}$, i.e. any morphism $F(X) \to F(Y)$ in $\mathcal{D}$ is of the form $F(f)$ for some $f : X \to Y$ in $\mathcal{C}$

- $H : \mathrm{Ab} \to \mathrm{Grp}$

Separable functors

Definition (Năstăsescu, Van den Bergh, Van Oystaeyen, 1989)

A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be separable if $\mathcal{F}$ splits, i.e. there is a natural transformation

$$\mathcal{P} : \mathrm{Hom}_{\mathcal{D}}(F-, F-) \to \mathrm{Hom}_{\mathcal{C}}(-,-)$$

such that

$$\mathcal{P} \circ \mathcal{F} = \mathrm{Id}_{\mathrm{Hom}_{\mathcal{C}}(-,-)}.$$

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In other words,

1. for any $f : X \to Y$ in $\mathcal{C}$, $\mathcal{P}_{X,Y} \circ \mathcal{F}_{X,Y}(f) = \mathcal{P}_{X,Y}(F(f)) = f$;
2. if $f, f' \in \mathcal{C}$ and $g, g' \in \mathcal{D}$ are such that

$$\begin{tikzcd}
F(X) \arrow[r, "g"] \arrow[d, "F(f)"'] \arrow[dr, phantom, "\circlearrowleft"] & F(X') \arrow[d, "F(f')"] & & X \arrow[r, "\mathcal{P}_{X,X'}(g)"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowleft"] & X' \arrow[d, "f'"] \\
F(Y) \arrow[r, "g'"'] & F(Y') & \Rightarrow & Y \arrow[r, "\mathcal{P}_{Y,Y'}(g')"'] & Y'
\end{tikzcd}$$

- If F is separable, then it is faithful.

Some properties

Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be functors.

- If F and G are separable, then $G \circ F$ is separable.
- If $G \circ F$ is separable, then F is separable.
- Let $f : X \to Y$ be a morphism in $\mathcal{C}$. If F is separable and $F(f)$ has a left (or right or two-sided) inverse $g \in \mathcal{D}$, then f has a left (or right or two-sided) inverse in $\mathcal{C}$, which is $\mathcal{P}_{X,Y}(g)$.

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$$\begin{tikzcd}
F(X) \arrow[r, "F(\mathrm{Id}_X)"] \arrow[d, "F(f)"'] \arrow[dr, phantom, "\circlearrowleft"] & F(X) \arrow[d, "F(\mathrm{Id}_X)"] \\
F(Y) \arrow[r, "g"'] & F(X)
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
X \arrow[r, "\mathcal{P}_{X,X}(F(\mathrm{Id}_X))"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowleft"] & X \arrow[d, "\mathrm{Id}_X"] \\
Y \arrow[r, "\mathcal{P}_{X,Y}(g)"'] & X
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\end{tikzcd}
$$

$\Rightarrow$ *Maschke's Theorem*: A short exact sequence which is split after we apply the separable functor F, is itself split.

Why *separable*?

Let $\varphi : R \to S$ be a ring homomorphism, and consider

$$\begin{tikzcd} \mathcal{M}_R \arrow[r, shift left, "\varphi^*=(-)\otimes_R S"] \arrow[r, phantom, "\perp"] & \mathcal{M}_S. \arrow[l, shift left, "\varphi_*"] \end{tikzcd}$$

Then,

- φ^* is separable $\Leftrightarrow$ φ splits as an R-bimodule map;
- φ_* is separable $\Leftrightarrow$ S/R is *separable*, i.e. the product map $m_S : S \otimes_R S \to S$ has an S-bimodule section σ (i.e. $m_S\sigma = \mathrm{Id}_S$).

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$\swarrow$

Let R be a commutative ring. A is a *separable* R-algebra if

$$\exists\ e = \sum_{i=1}^{n} x_i \otimes_R y_i \ \in A \otimes_R A$$

for some $x_i, y_i \in A$ such that $\sum_{i=1}^{n} x_i y_i = 1_A$ and $\forall h \in A,\ \sum_{i=1}^{n} hx_i \otimes_R y_i = \sum_{i=1}^{n} x_i \otimes_R y_i h$.

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- $\mathbb{C}/\mathbb{R}$ is separable: $e = \frac{1}{2}(1 \otimes 1 - i \otimes i)$
- $\mathbb{R}/\mathbb{Q}$ is not separable
- $M_n(R)$ is a separable R-algebra: $e = \sum_{i=1}^n e_{i,1} \otimes_R e_{1,i}$, where $(e_{i,j})_{(i,j)} = 1_R$ and $(e_{i,j})_{(h,k)} = 0_R$ for $(h,k) \neq (i,j)$

Rafael's Theorem, 1990

Let $\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, shift left, "G"] \end{tikzcd}$ be an adjoint pair of functors with unit η and counit ϵ. Then,

1. F is separable if and only if the unit $\eta : \mathrm{Id}_{\mathcal{C}} \to GF$ splits, i.e. there exists a natural transformation $\nu : GF \to \mathrm{Id}_{\mathcal{C}}$ such that $\nu \circ \eta = \mathrm{Id}_{\mathrm{Id}_{\mathcal{C}}}$;
2. G is separable if and only if the counit $\epsilon : FG \to \mathrm{Id}_{\mathcal{D}}$ cosplits, i.e. there exists a natural transformation $\gamma : \mathrm{Id}_{\mathcal{D}} \to FG$ such that $\epsilon \circ \gamma = \mathrm{Id}_{\mathrm{Id}_{\mathcal{D}}}$.

Rafael's Theorem, 1990

Let $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ ($F \dashv G$) be an adjoint pair of functors with unit η and counit ϵ. Then,

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Who is Rafael?

M. Saorin (Univ. of Murcia, Spain)
D. Herbera (Univ. Autonoma of Barcelona, Spain)
R. Colpi (Univ. of Padova, Italy)
A. Del Rio Mateos (Univ. of Murcia, Spain)
F. Van Oystaeyen (UIA, Univ. of Antwerp, Belgium)
A. Giaquinta (Univ. of Pennsylvania, USA)
E. Gregorio (Univ. of Padova, Italy)
L. Bionda (Univ. of Padova, Italy)

Separable $=$ "naturally faithful" $\Longrightarrow$ FAITHFUL

? $\Longrightarrow$ FULL

Naturally full functors

Definition (Ardizzoni, Caenepeel, Menini, Militaru, 2006)

A functor $F : \mathcal{C} \to \mathcal{D}$ is called naturally full if there exists a natural transformation

$$\mathcal{P} : \mathrm{Hom}_{\mathcal{D}}(F-, F-) \to \mathrm{Hom}_{\mathcal{C}}(-,-)$$

such that

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$$\mathcal{P}_{V,T}(F(h) \circ f \circ F(k)) = h \circ \mathcal{P}_{X,Y}(f) \circ k$$

$\forall f : F(X) \to F(Y)$ in $\mathcal{D}$ and $\forall k : V \to X$, $h : Y \to T$ in $\mathcal{C}$.

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$\forall f : F(X) \to F(Y)$ in $\mathcal{D}$ and $\forall k : V \to X$, $h : Y \to T$ in $\mathcal{C}$.

- If F is naturally full, then it is full.
- If F and G are naturally full, then $G \circ F$ is naturally full.
- If $G \circ F$ is naturally full and G is faithful, then F is naturally full.

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- If F and G are naturally full, then $G \circ F$ is naturally full.
- If $G \circ F$ is naturally full and G is faithful, then F is naturally full.

Remark

F is fully faithful $\Leftrightarrow$ F is separable and naturally full

Rafael type Theorem

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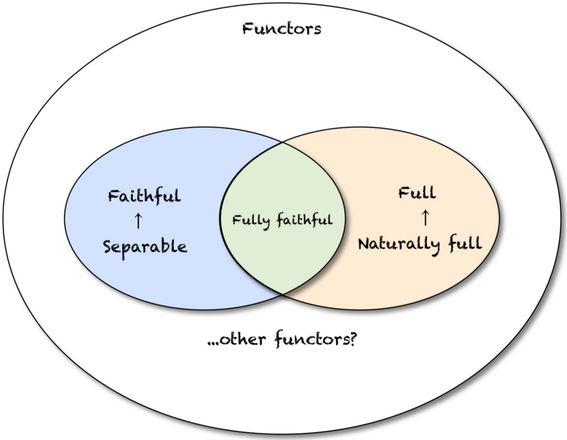
1. F is naturally full if and only if the unit $\eta : \mathrm{Id}_{\mathcal{C}} \to GF$ cosplits, i.e. there is $\nu : GF \to \mathrm{Id}_{\mathcal{C}}$ such that $\eta_C \circ \nu_C = \mathrm{Id}_{GFC}$, for all $C \in \mathcal{C}$;
2. G is naturally full if and only if the counit $\epsilon : FG \to \mathrm{Id}_{\mathcal{D}}$ splits, i.e. there is $\gamma : \mathrm{Id}_{\mathcal{D}} \to FG$ such that $\gamma_D \circ \epsilon_D = \mathrm{Id}_{FGD}$, for all $D \in \mathcal{D}$.

Example

$$\mathcal{M}_R \underset{\varphi_*}{\overset{\varphi^*=(-)\otimes_R S}{\rightleftarrows}} \mathcal{M}_S$$

- φ_* is naturally full $\Leftrightarrow$ it is full.
- φ^* is naturally full $\Leftrightarrow$ there exists $E \in {}_R\mathrm{Hom}(S, R)_R$ such that $\varphi \circ E = \mathrm{Id}_S$.

Further investigations



Let $F : \mathcal{C} \to \mathcal{D}$ and $H : \mathcal{C} \to \mathcal{E}$ be functors.

- *Separable functors of II type*: F is *H-separable*, if

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(-,-) \arrow[r, "\mathcal{F}"] \arrow[d, "\mathcal{H}"'] & \mathrm{Hom}_{\mathcal{D}}(F-,F-) \arrow[dl, dotted, "\exists\, \mathcal{P}"] \\ \mathrm{Hom}_{\mathcal{E}}(H-,H-) & \end{tikzcd}$$

  such that $\mathcal{P} \circ \mathcal{F} = \mathcal{H}$.

- F is *heavily-separable* (2020) if it is separable and for every $X, Y, Z \in \mathcal{C}$,

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{D}}(FX,FY) \times \mathrm{Hom}_{\mathcal{D}}(FY,FZ) \arrow[r, "\mathcal{P}_{X,Y} \times \mathcal{P}_{Y,Z}"] \arrow[d, "\circ"'] & \mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathrm{Hom}_{\mathcal{C}}(Y,Z) \arrow[d, "\circ"] \\ \mathrm{Hom}_{\mathcal{D}}(FX,FZ) \arrow[r, "\mathcal{P}_{X,Z}"'] & \mathrm{Hom}_{\mathcal{C}}(X,Z) \end{tikzcd}$$

  commutes, i.e. on elements $\mathcal{P}_{X,Z}(f \circ g) = \mathcal{P}_{Y,Z}(f) \circ \mathcal{P}_{X,Y}(g)$.

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