Billiards with refraction: an example from Celestial Mechanics

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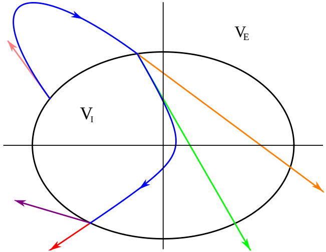
Dynamical model

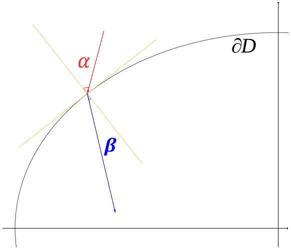
Let $D \subset \mathbb{R}^2$ be a regular domain, $0 \in D^\circ$, and consider the orbits with zero energy subjected to the potential

$$V(z) = \begin{cases} V_I(z) = \mathcal{E} + h + \frac{\mu}{|z|} & \text{if } z \in D, \\ V_E(z) = \mathcal{E} - \frac{\omega^2}{2}|z|^2 & \text{if } z \notin \bar{D}, \end{cases}$$

with $\mathcal{E}, h, \mu, \omega > 0$, while on the boundary ∂D the following junction rule (*Snell's law*-type) holds:

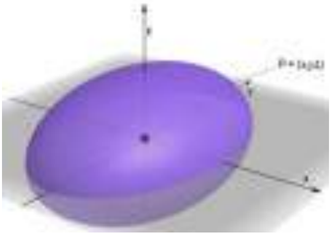
$$\sqrt{V_E(z)} \sin \alpha = \sqrt{V_I(z)} \sin \beta$$





Motivations - Black Hole in an elliptical Galaxy

Suppose to have an ellipsoidal galaxy with constant density and a Black Hole at its center.



$$\begin{cases} V_{Gal}(P) = -\frac{\omega_x^2}{2}x^2 - \frac{\omega_y^2}{2}y^2 - \frac{\omega_z^2}{2}z^2 + C_G \\ V_{BH}(P) = \frac{\mu}{\sqrt{x^2+y^2+z^2}} + C_{BH} \end{cases}$$

$\omega_x, \omega_y, \omega_z > 0,\ C_G, C_{BH} \in \mathbb{R}$.

BH's region of influence$= \tilde{D} = \{P \in \mathbb{R}^3 \mid |V_{BH}(P)| \gg |V_{Gal}(P)|\}$

π_{xy} invariant under the dynamics: if $\omega_x = \omega_y$ and $0 < C_G < C_{BH}$, setting $D = \tilde{D} \cap \pi_{xy}$ we can use our 2-D model to study the motion of P on π_{xy}.

First return map

$\partial D = supp(\gamma)$ with $\gamma : \mathbb{R}_{/2\pi\mathbb{Z}} \to \mathbb{R}^2$ regular closed curve; if $(p_0, v_0) \in \partial D \times \mathbb{R}^2$ are the initial conditions of an outward-pointing orbit, there are $\xi_0 \in \mathbb{R}_{/2\pi\mathbb{Z}}, \alpha_0 \in [-\pi/2, \pi/2]$ such that, if $\hat{t}(\xi), \hat{n}(\xi)$ are the tangent and inner normal unit vectors:

$$p_0 = \gamma(\xi_0), v_0 = \sqrt{2V_E(p_0)}(\sin\alpha_0 \hat{t}(\xi_0) + \cos\alpha_0 \hat{n}(\xi_0))$$

$\Rightarrow$ the pair (ξ_0, α_0) completely determines the initial conditions on the boundary ∂D. We define the *first return map* $F : \mathbb{R}_{/2\pi\mathbb{Z}} \times [-\frac{\pi}{2}, \frac{\pi}{2}] \to \mathbb{R}_{/2\pi\mathbb{Z}} \times [-\frac{\pi}{2}, \frac{\pi}{2}]$

$$(\xi_0, \alpha_0) \xrightarrow[\text{refraction E-I}]{\text{outer arc}} (\tilde{\xi}, \tilde{\alpha}) \xrightarrow[\text{refraction I-E}]{\text{inner arc}} (\xi_1, \alpha_1)$$

Variational approach

Define the generating fuction

$$S : \mathbb{R}_{/2\pi\mathbb{Z}} \times \mathbb{R}_{/2\pi\mathbb{Z}} \quad S(\xi_0, \xi_1) = S_E(\xi_0, \tilde{\xi}) + S_I(\tilde{\xi}, \xi_1)$$

$$\begin{aligned} S_{E\backslash I}(a,b) &= d_{E\backslash I}(\gamma(a), \gamma(b)) = \\ &= \min \left\{ \begin{array}{r} \int_0^1 |\dot{\lambda}(t)| \sqrt{V_{E\backslash I}(\lambda(t))} dt \mid \lambda(t) \text{ piecewise differentiable} \\ \text{s.t. } \lambda(0) = \gamma(a),\ \lambda(1) = \gamma(b) \end{array} \right\} \end{aligned}$$

and $\tilde{\xi}$ such that, fixed ξ_0, ξ_1, $\partial_\xi (S_E(\xi_0, \xi) + S_I(\xi, \xi_1))_{|\xi = \tilde{\xi}} = 0$.

$$\begin{aligned} \text{Conjugated actions: } I_0 &= -\partial_{\xi_0} S(\xi_0, \xi_1) = \sqrt{V_E(\gamma(\xi_0))} \sin \alpha_0 \\ I_1 &= \partial_{\xi_1} S(\xi_0, \xi_1) = \sqrt{V_E(\gamma(\xi_1))} \sin \alpha_1 \end{aligned}$$

$$\mathcal{F} : \mathbb{R}_{/2\pi\mathbb{Z}} \times \left(-\sqrt{\mathcal{E} - \frac{\omega^2}{2}}, \sqrt{\mathcal{E} - \frac{\omega^2}{2}} \right) \to \mathbb{R}_{/2\pi\mathbb{Z}} \times \left(-\sqrt{\mathcal{E} - \frac{\omega^2}{2}}, \sqrt{\mathcal{E} - \frac{\omega^2}{2}} \right)$$

$$(\xi_0, I_0) \mapsto (\xi_1, I_1)$$

When $S(\xi_0, \xi_1)$ is well defined, $\mathcal{F}$ is **conservative**.

Circular case

If D is a disk of radius 1, $\mathcal{F}$ is a **shift map** $(\xi_0, I_0) \mapsto (\xi_1, I_1) = (\xi_0 + \theta(I_0), I_0)$, where

$$\theta(I) = \begin{cases} \arctan\left(\frac{\mathcal{E}-2I^2}{I\sqrt{4\mathcal{E}-2(2I^2+\omega^2)}}\right) + 2\arccos\left(\frac{2I^2-\mu}{\sqrt{4(\mathcal{E}+h)I^2+\mu^2}}\right) - 2\pi & \text{if } I > 0 \\ 0 & \text{if } I = 0 \\ \arctan\left(\frac{\mathcal{E}-2I^2}{I\sqrt{4\mathcal{E}-2(2I^2+\omega^2)}}\right) - 2\arccos\left(\frac{2I^2-\mu}{\sqrt{4(\mathcal{E}+h)I^2+\mu^2}}\right) + \pi & \text{if } I < 0 \end{cases}$$

Proposition

There is $\tilde{\mathcal{I}} \subset \mathcal{I} = \left(-\sqrt{\mathcal{E} - \frac{\omega^2}{2}}, \sqrt{\mathcal{E} - \frac{\omega^2}{2}}\right)$, $|\tilde{\mathcal{I}}| \leq 10$, such that for every $\xi_0 \in \mathbb{R}_{/2\pi\mathbb{Z}}$, $I_0 \in \mathcal{I} \backslash \tilde{\mathcal{I}}$:

- *$\mathcal{F}$ is well defined and conservative;*
- *$\mathcal{F}$ satisfies the **twist condition***

$$\frac{\partial \xi_1}{\partial I_0}(\xi_0, I_0) \neq 0.$$

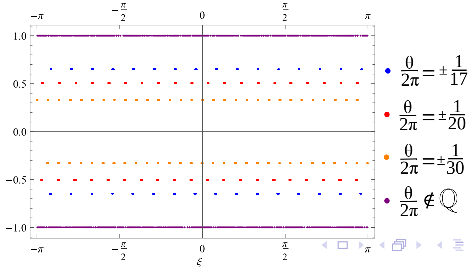
Orbits on the circle

Orbit: $\{(\xi_k, I_k)\} = \{\mathcal{F}^k(\xi_0, I_0)\}_{k\in\mathbb{Z}}$.
Rotation number: $\rho(\xi_0, I_0) = \lim_{k\to\infty} \frac{\xi_k}{k} = \theta(I_0)$ (circular case).
If $\theta(I_0) = 2\pi\dfrac{p}{q},\ p, q \in \mathbb{Z} \Longrightarrow (\xi_q, I_q) = (\xi_0 + 2\pi p, I_0) \equiv (\xi_0, I_0)$
$\Longrightarrow (\xi_0, I_0)$ is (p, q) - periodic.

Theorem

$\exists C = C(\mathcal{E}, \omega, h, \mu) > 0$ *such that for every* $\rho \in (-C, C)$ *except for a finite number of values* $\exists I_0^{\pm} \in \mathcal{I}\backslash\tilde{\mathcal{I}}$ *s.t. for every* $\xi_0 \in \mathbb{R}_{/2\pi\mathbb{Z}}$
$\rho(\xi_0, I_0^{\pm}) = \rho$.

$\mathscr{E} = 7,\ \omega^2 = 3$
$h = 2,\ \mu = 15$
$C \approx 0.391$



General case: homotetic fixed point

Suppose that $\gamma : \mathbb{R}_{/2\pi\mathbb{Z}} \to \mathbb{R}^2$ is a regular closed curve, $supp(\gamma) = \partial D$, $0 \in D^\circ$.

Proposition

If $\bar{\xi} \in \mathbb{R}_{/2\pi\mathbb{Z}}$ is such that $\dot{\gamma}(\bar{\xi}) \perp \gamma(\bar{\xi}) \Rightarrow (\bar{\xi}, 0)$ is a fixed point for $\mathcal{F}$, corresponding to the homotetic solution of initial conditions

$$(p_0, v_0) = \left(\gamma(\bar{\xi}), \frac{\sqrt{2V_E(\gamma(\bar{\xi}))}\gamma(\bar{\xi})}{|\gamma(\bar{\xi})|} \right)$$

The linear stability of $(\bar{\xi}, 0)$ can be studied by computing the Jacobian matrix of $\mathcal{F}$.

Variational methods + Levi-Civita regularization + Implicit function theorem

$$\downarrow$$

$$D\mathcal{F}(\bar{\xi}, 0) = \begin{pmatrix} \frac{\partial \xi_1}{\partial \xi_0}(\bar{\xi}, 0) & \frac{\partial \xi_1}{\partial I_0}(\bar{\xi}, 0) \\ \frac{\partial I_1}{\partial \xi_0}(\bar{\xi}, 0) & \frac{\partial I_1}{\partial I_0}(\bar{\xi}, 0) \end{pmatrix} \rightsquigarrow \lambda_1, \lambda_2 \text{ eigenvalues}$$

Homotetic fixed points - Linear stability

Proposition

Let $p(\lambda)$ be the characteristic polynomial of $D\mathcal{F}(\bar{\xi},0)$, and Δ its discriminant.
Since $\det\left(D\mathcal{F}(\bar{\xi},0)\right)=1$, *then:*

- $\Delta>0 \Rightarrow \lambda_1,\lambda_2\in\mathbb{R}, \lambda_2=1/\lambda_1, \lambda_1\neq\lambda_2 \Rightarrow |\lambda_1|>1, |\lambda_2|<1$, *then* $(\bar{\xi},0)$ *is an unstable saddle point;*
- $\Delta<0 \Rightarrow \lambda_1,\lambda_2\in\mathbb{C}\backslash\mathbb{R}, \lambda_1=\bar{\lambda}_2, \lambda_1\neq\lambda_2 \Rightarrow |\lambda_1|=|\lambda_2|=1$ *and* $(\bar{\xi},0)$ *is a stable center point;*
- $\Delta=0 \rightarrow$ *degenerate case (e.g. circular domain).*

The discriminant Δ depends on the physical parameters $\mathcal{E},h,\mu,\omega$ and on γ and $\bar{\xi}$ trough $|\gamma(\bar{\xi})|$, $|\dot{\gamma}(\bar{\xi})|$ and $k(\bar{\xi})$, i.e. the curvature of γ in $\bar{\xi}$: fixed $\bar{\xi}$ such that $(\bar{\xi},0)$ is a homotetic fixed point,changing the values of the physical parameters may modify its stability properties$\rightarrow$**bifurcations**.

Elliptic case - Bifurcation for μ

Suppose that ∂D is an ellipse parametrized by $\gamma(\xi) = (a\cos\xi, b\sin\xi)$, $a = 1,\ b = a\sqrt{1-e^2},\ 0 \leq e < 1 \Rightarrow (0,0)$ and $(\pi/2, 0)$ are fixed points for $\mathcal{F}$.

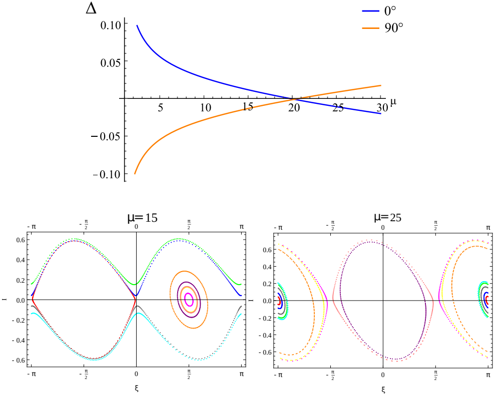
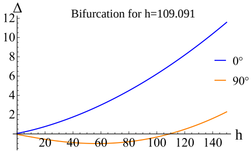


Figure: $\mathcal{E} = 2.5, h = 2, \omega = \sqrt{2}, e = 0.1$. Up: value of Δ for $(0,0)$ and $(\pi/2, 0)$ and $\mu \in [1, 30]$. Down: Poincaré sections for different values of μ, 200 points.

Elliptic case - Bifurcation for h



The stability behaviour of the equilibria suggests the presence of a pitchfork bifurcation, with the arising of a new equilibrium point between 0 and $\pi/2$.

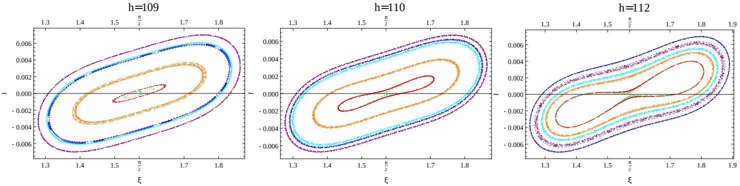
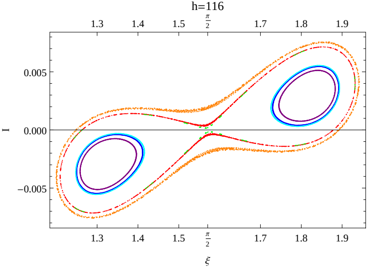
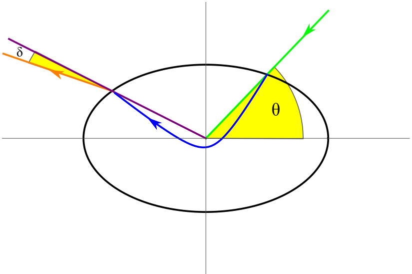


Figure: $\mathcal{E} = 2.5, \mu = 2, \omega = \sqrt{2}$, $e = 0.1$. Up: value of Δ for $(0, 0)$ and $(\pi/2, 0)$ and $h \in [0, 150]$. Down: Poincaré sections for different values of h, 2000 points.

Elliptic case - Search for new equilibria





If $\lambda_{1/2}^{(k)}$ are the eigenvalues of $D\mathcal{F}^k(\bar{\xi}, 0) \Rightarrow \lambda_{1/2}^{(k)} = \lambda_{1/2}^k \Rightarrow$ the bifurcation values are the same for *all the iterates*.

The Poincaré sections show the arising of an orbit of minimal period 2 for $h >$ bifurcation value.

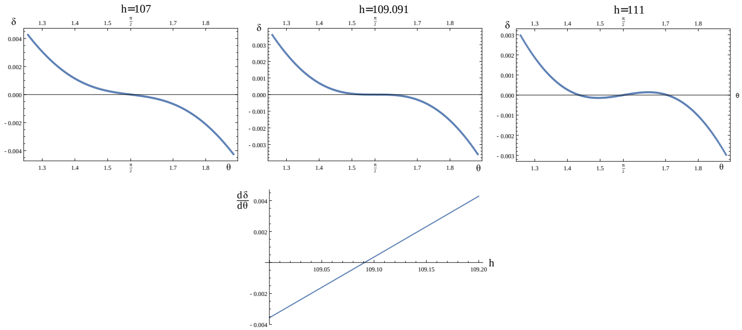
Idea: search for 2-periodic **brake orbits** via the **shooting method**.

We search for 2-periodic orbits which are homotetic in their outer arcs: consider the free fall map

$$\Phi : [0, 2\pi] \to [0, 2\pi], \quad \theta \mapsto \delta$$

Elliptic case - Search of new equilibria

If $\delta = \Phi(\theta) = 0$, the outer branches are both homotetic, and the whole orbit is $2-$periodic.



Up: plot of $\delta = \Phi(\theta)$ in a neighborhood of $\pi/2$ for different values of h.
Down: first derivative of $\delta^{\pi/2} = \Phi(\pi/2)$ as a function of h.
Conclusion: the 2-periodic orbits which appear when $h >$bifurcation value are brake.

Further research

- *Elliptic case:* systematic study of the orbits near to the homotetics for every value of the eccentricity;

- *General case:* perturbative methods on the circular case, with a small deformation of the boundary ∂D. In particular:

  - **KAM theory**: existence of quasi-periodic invariant tori with diophantine rotation number for the perturbed system;

  - **Mather theory** (Poincaré Birkhoff theorem): existence of invariant orbits with prescribed rotation number, under the hypothesis of *twist condition*.

Bibliography

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