Origami and Mathematical thinking

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Origami and Mathematics Education

While the **relationship** between origami and mathematics is studied, **research in mathematics education** on this relationship is **limited**. Few studies have examined the use of origami as a teaching tool to develop **visual and spatial skills**, or a **sense of space**.

Low attention to the thinking processes, even though some studies give merit to the practice of origami in the classroom:

- improvement of geometric reasoning, visual perception, psychomotor skills, manipulation, imagination and creativity;
- active involvement in the creative process;
- motivation in formulating and evaluating conjectures on geometric relationships;
- making the students approach the concept of axiomatic system.

Spatial visualization

Manipulation

Mathematical language

Creativity

Conjecturing

Abstract thinking

[Boakes, 2009; Çakmak, Isiksal, Koc, 2014; Maffini, 2020]

What to investigate?

Involved **processes**:

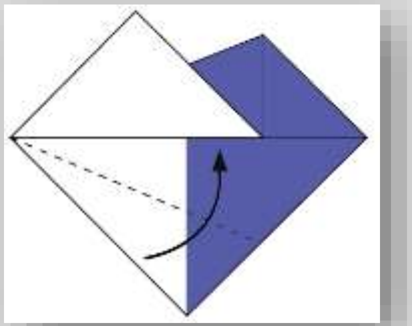
- Imagination
- Visualization
- Prevision
- Argumentation
- Creativity

Involved **activities**:

- Manipulation
- Movement
- Folding
- Material activity

The **role** played by:

- Diagrams
- Paper
- Body
- Language
- Technology

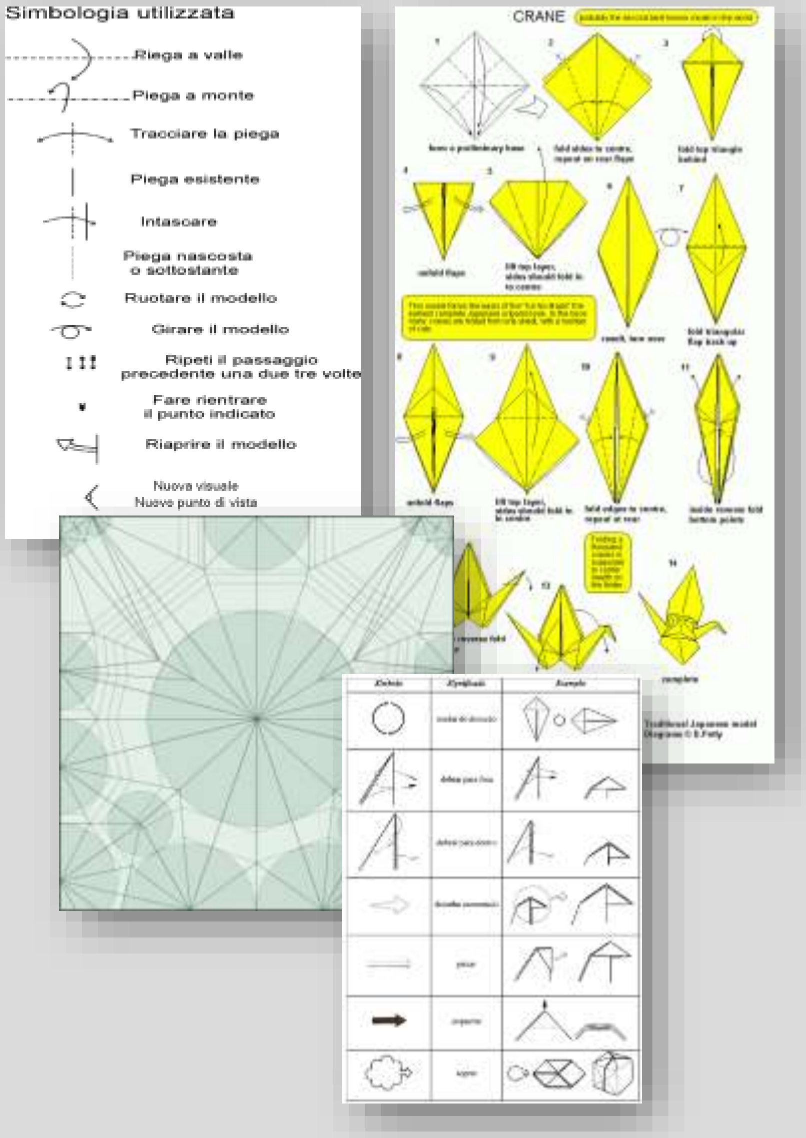


The **relationship** between:

- Diagrams and origami
- Plane and space
- Origami and mathematical properties

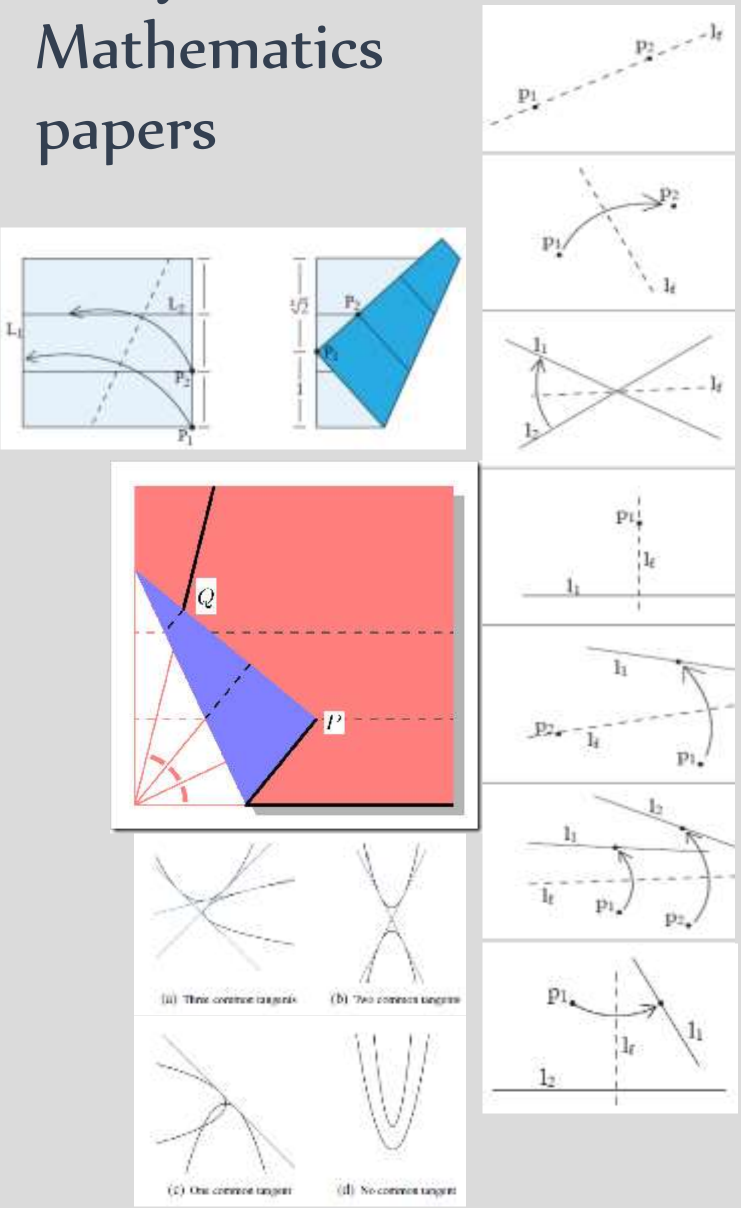
Origami and Mathematics Education

Analysis of Origami books

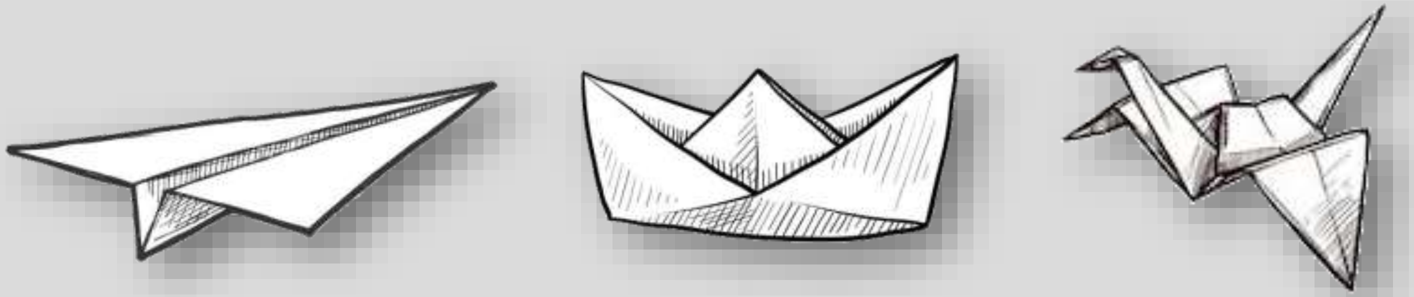


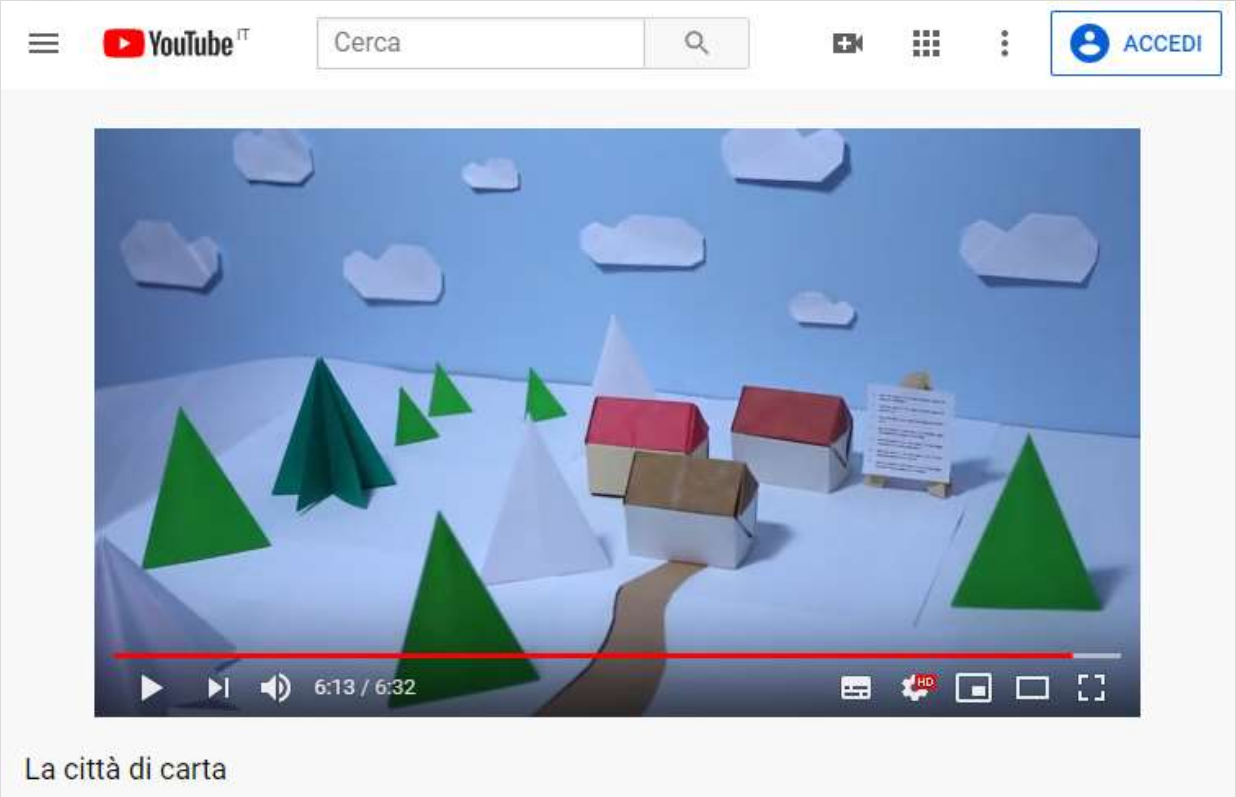
Analysis of Research papers

Analysis of Mathematics papers



My interest in origami





CENTRO DIFFUSIONE ORIGAMI

Origami
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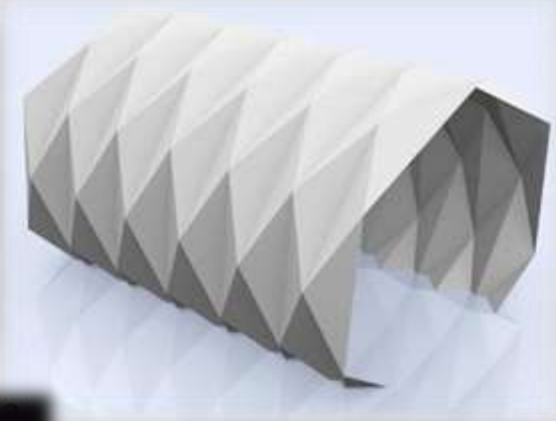
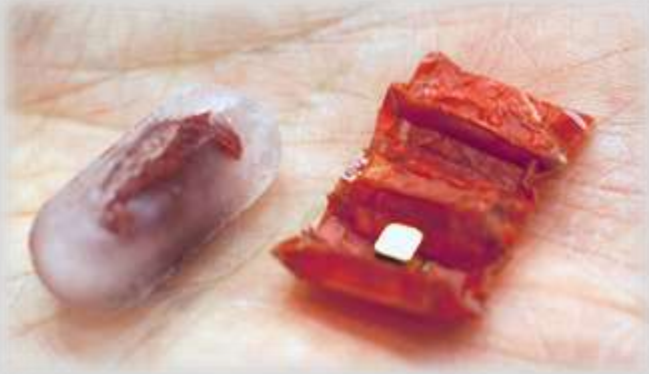
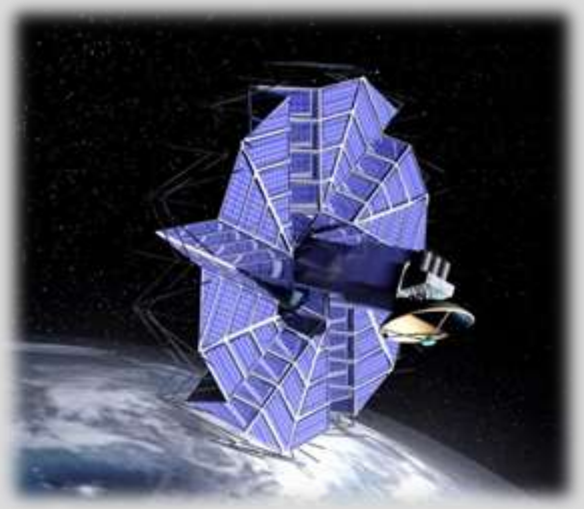
My interest in origami

Origami's interdisciplinary relevance:

- Mathematics
- Technology
- Science
- Engineering
- Education
- Art
- Medicine
- Architecture
- History



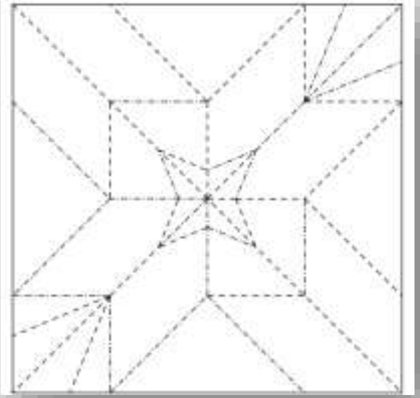


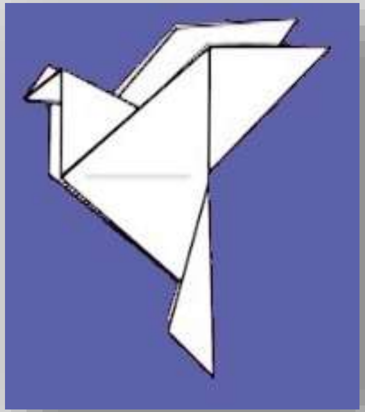


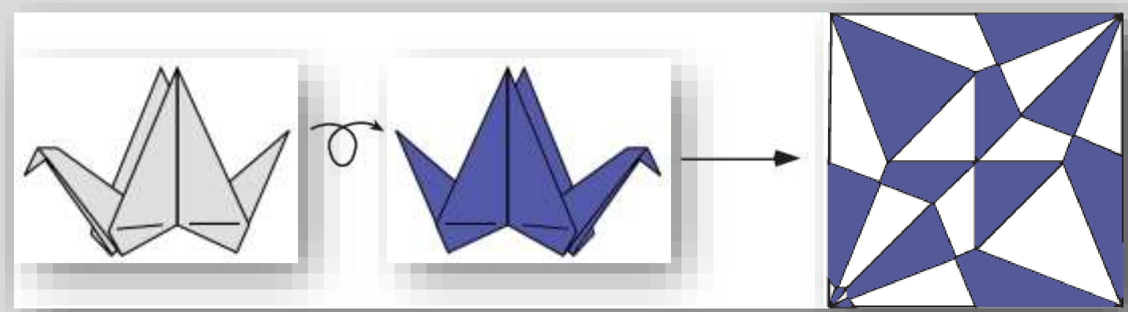
Origami and Mathematics

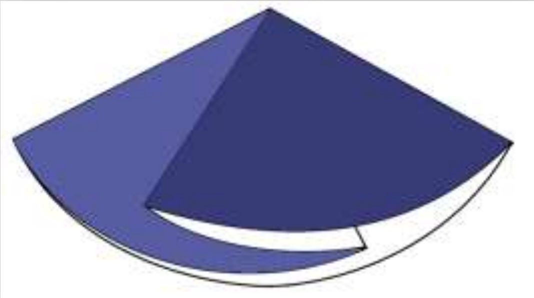
The relevance of paper folding has been investigated in relation to many mathematical contents, like:

- Geometry and Figures
- Plane/Space transformations
- Axioms and relations with Euclidean geometry
- Origami/Crease-pattern relations
- Flat Foldability and Colorability
- Construction Algorithms



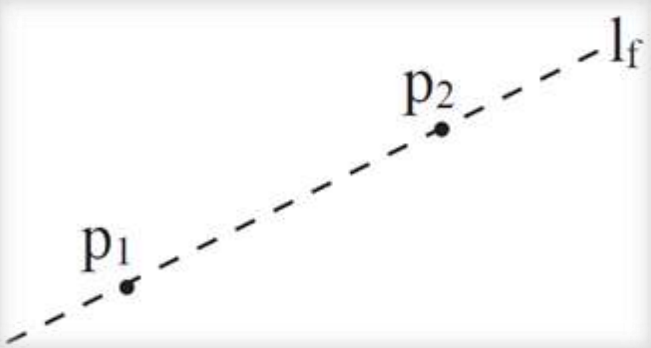




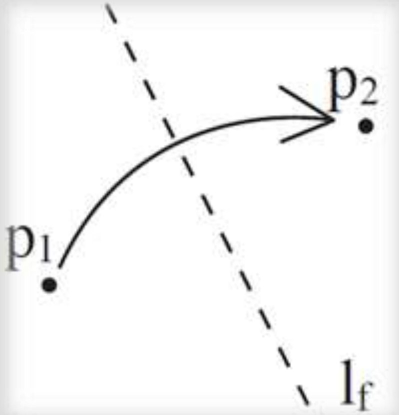


Origami geometry Axioms

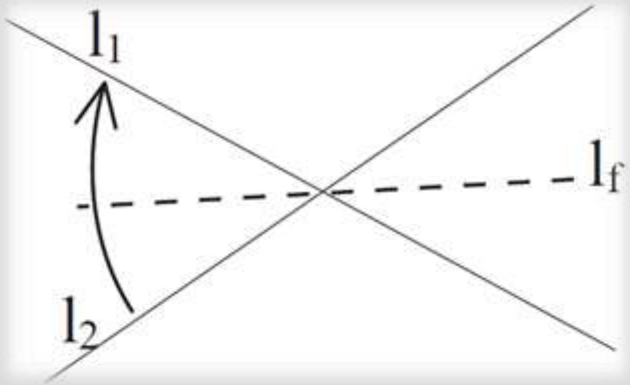
In 1989 Scimemi, Huzita, Justin (and then Hatori) identified seven different ways one could create a single crease by aligning one or more combinations of points and lines, known as the *Huzita-Justin* or *Huzita-Hatori Axioms*.



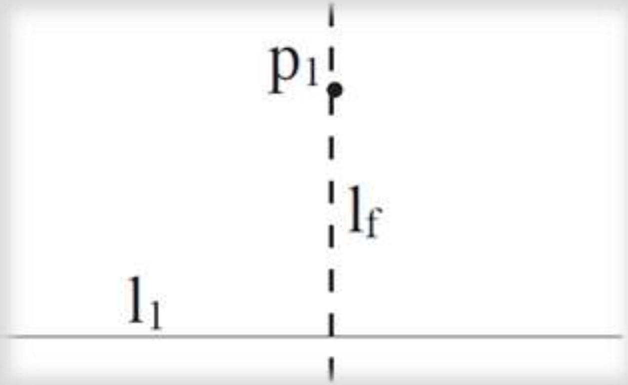
1. Given two points we can fold a line that connect them.



2. Given two points we can fold one point onto the other.



3. Given two lines we can fold one line onto the other.



4. Given a point and a line, we can make a fold perpendicular to the line passing through the point.

[Huzita, 1989; Alperin & Lang, 2009]

Origami geometry Axioms

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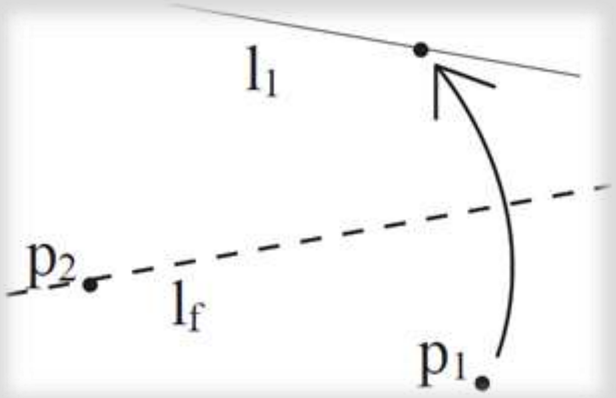
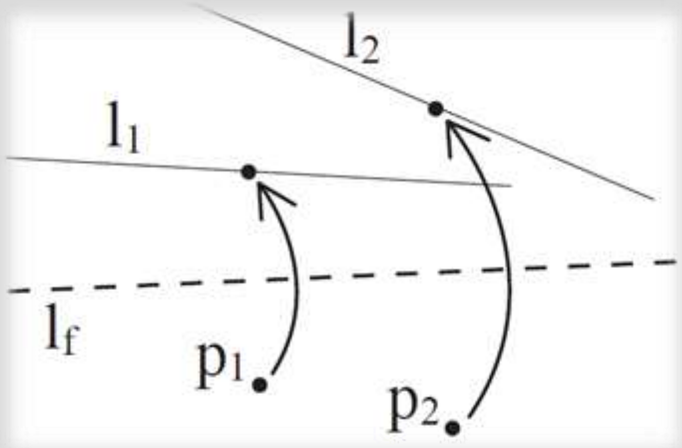
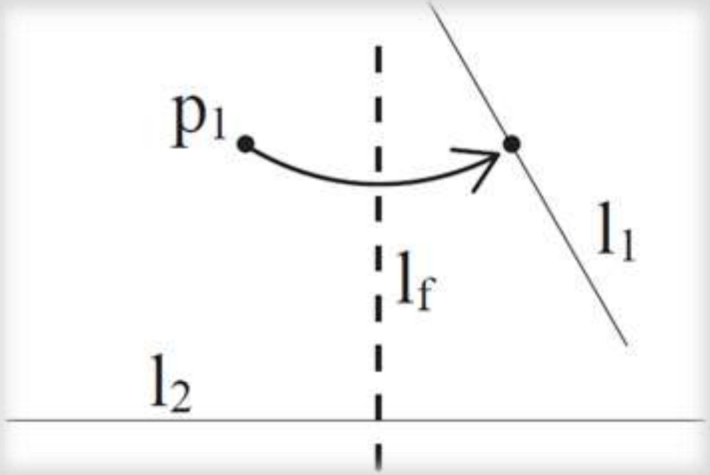






5. Given two points and a line, we can make a fold that places one point onto the line and passes through the point.

6. Given two points and two lines, we can make a fold that places one point onto one line and the other point onto the other line.

7. Given a point and two lines, we can make a fold perpendicular to a line that places the point onto the other line.

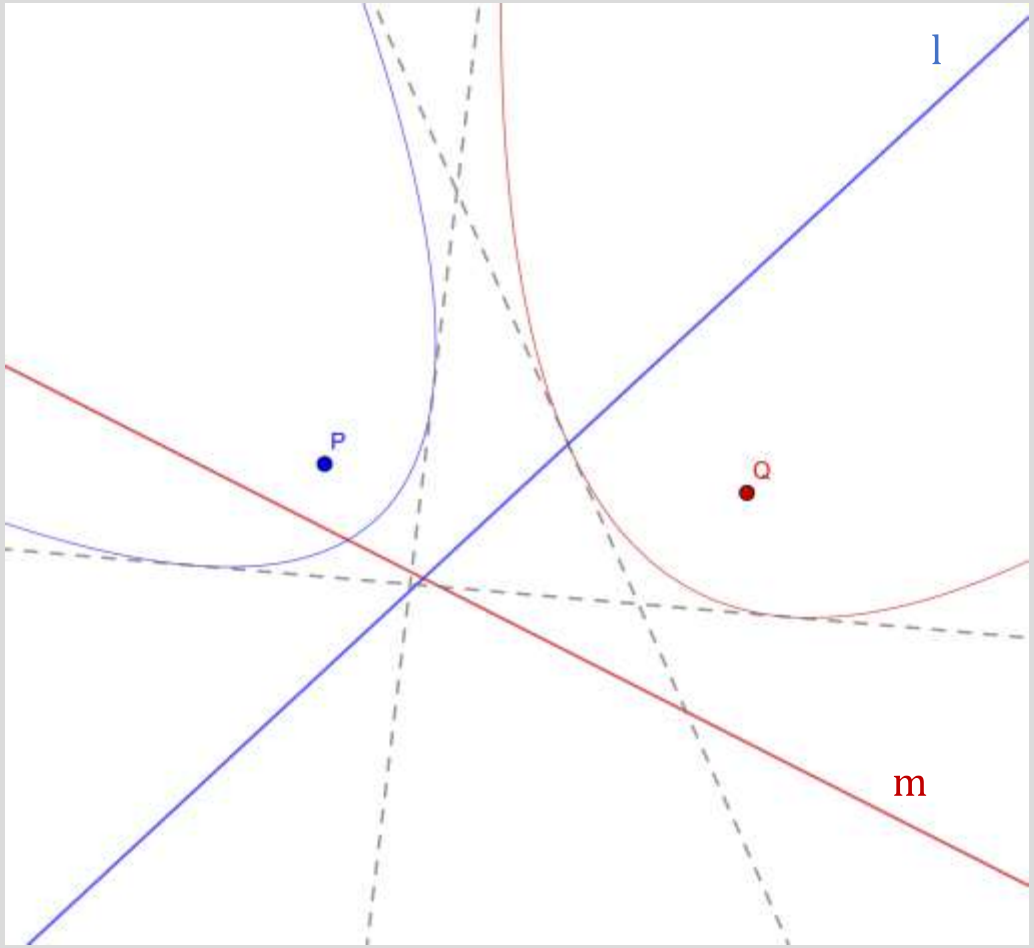
[Huzita, 1989; Alperin & Lang, 2009]

Origami geometry Axioms

In 1989 Scimemi, Huzita, Justin (and then Hatori) identified seven different ways one could create a single crease by aligning one or more combinations of points and lines, known as the *Huzita-Justin* or *Huzita-Hatori Axioms*.

Theorem. The list of Huzita-Justin axioms is complete, i.e. there are no more possible alignments with origami with just one fold each time. (Alperin & Lang 2009)

The sixth axiom



Axiom 6: Given two points P and Q and two lines l and m, we can make a fold that place P onto l and Q onto m.

↓

At most three folds

↓

Three common tangents to two parabolas

↓

Third-degree problem

Margherita Piazzolla Beloch

«si può osservare che questo metodo [del ripiegamento della carta] più che una semplice curiosità matematica, costituisca uno strumento che può servire utilmente per la **risoluzione effettiva di una vasta categoria di problemi geometrici non risolubili con riga e compasso**, come pure può rappresentare un effettivo **risparmio di tempo** per certe costruzioni risolubili con riga e compasso»

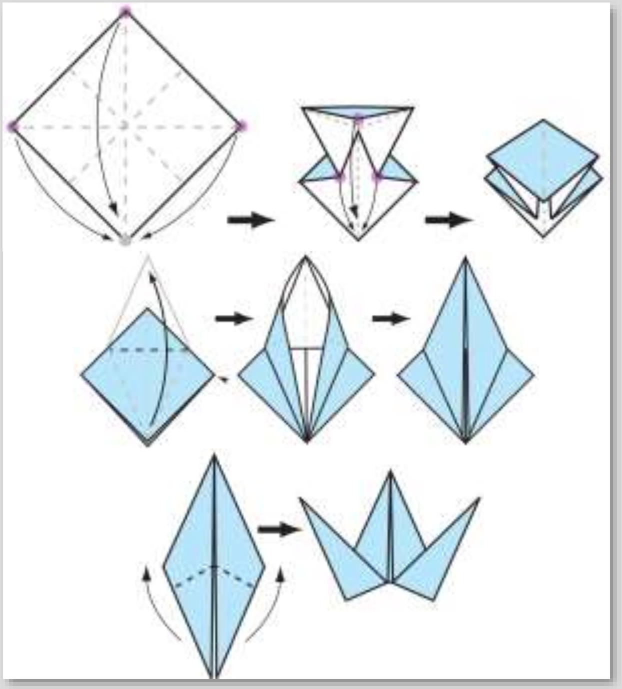
(Beloch 1936)

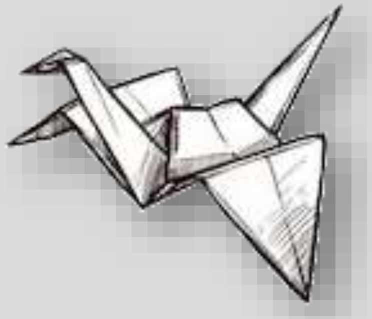
The paper folding method allows:

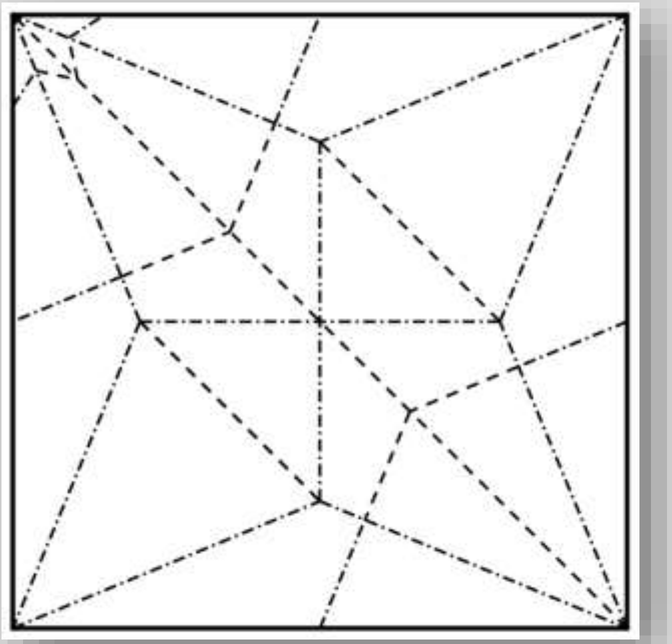
- To make some geometric constructions in an easier way than ruler and compass (perpendicular, conics' tangents, …)
- **To solve geometric problems unsolvable with ruler and compass:**
  - **angle trisection**
  - **cube duplication**

Origami/Crease-pattern relation

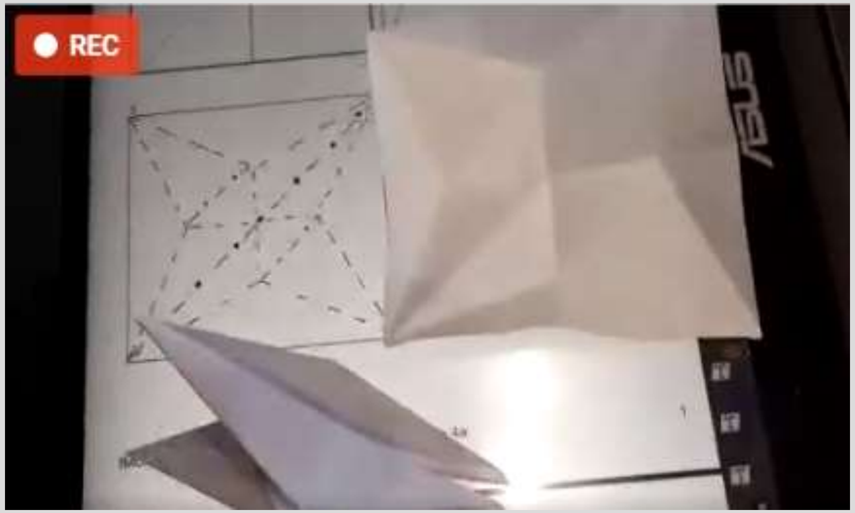
Definition. The **crease-pattern** of an origami is the plane diagram which consists of the lines that represent the **fundamental** valley and mountain folds, i.e. all and only the folds that are used to close the origami (in its final form).





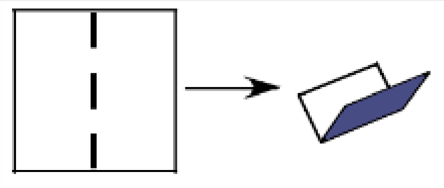


Experimentation with University students

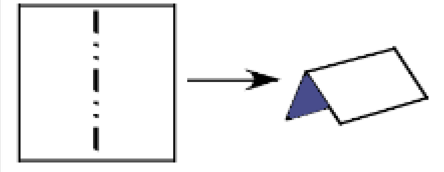


Flat Foldability and Colorability

Maekawa's theorem (1980).
Let M denote the number of mountain creases in the crease-pattern of a flat origami, and V be the number of valley creases. Then M – V = ±2.

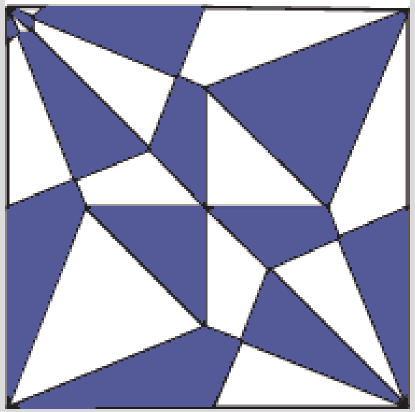


valley fold

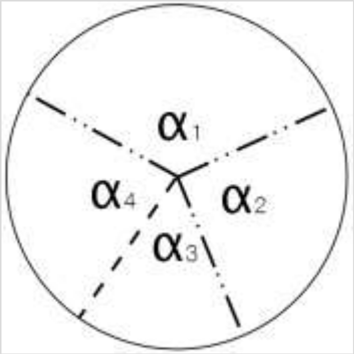


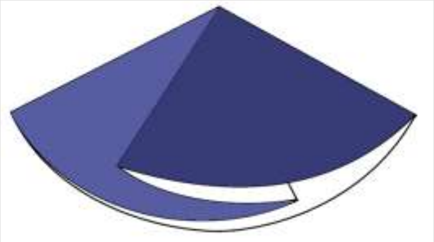
mountain fold

Corollary.
If we think of an origami crease-pattern as a graph, then every flat origami crease-pattern is 2 face-colorable.



Kawasaki's theorem (1980).
The sum of the alternate angles about the vertex in the crease-pattern of a flat origami is π.





$$\alpha_1 + \alpha_3 = \alpha_2 + \alpha_4 = \pi$$

[Hull, 1994]

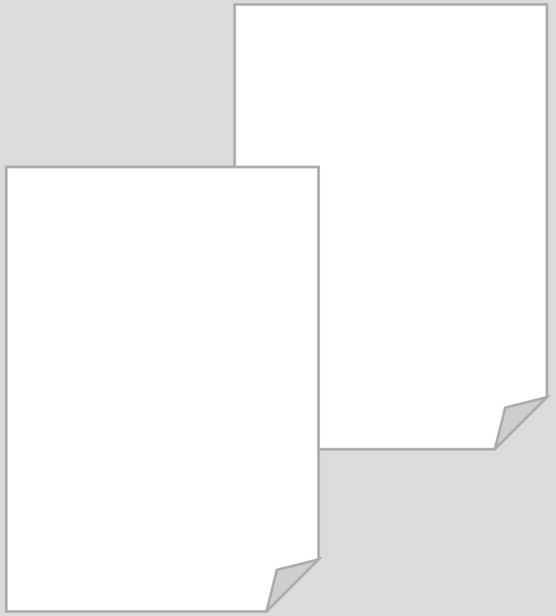
Essential bibliography

- Alperin, R.C. & Lang, R.J. (2006). One-, two-, and multi-fold origami axioms. In R.J. Lang (Ed.), *Proceedings of the Fourth International Meeting of Origami in Science, Mathematics, and Education* (pp. 371–393).
- Beloch, M. P. (1936). Sul metodo del ripiegamento della carta per la risoluzione dei problemi geometrici. *Periodico di Matematiche*, 16:4, 104-108.
- Boakes, N. J. (2009). Origami instruction in the middle school mathematics classroom: Its impact on spatial visualization and geometry knowledge of students. *Research in Middle Level Education Online*, 32:7, 1-12.
- Çakmak, S., Isiksal, M. & Koc, Y. (2014). Investigating effect of origami-based instruction on elementary students' spatial skills and perceptions. *The Journal of Educational Research*, 107:1, 59-68.
- Hull, T. (1994). On the Mathematics of Flat Origamis. *Congressus Numerantium* 100, 215-224.
- Huzita, H. (1989). Axiomatic Development of Origami Geometry. In H. Huzita (Ed.), *Proceedings of the First International Meeting of Origami Science and Technology* (pp. 143–158).
- Maffini A. (2020). La geometria dell'origami. *Didattica della matematica. Dalla ricerca alle pratiche in aula,* 2020 (8), 92-114.
- OSME Proceedings. International Conference on Origami Science, Mathematics, and Education (1989, 1994, 2001, 2006, 2010, 2014, 2018)
- Scimemi, B. (2012). Geometria sintetica. Padova: Coop. Libraria Editrice Università di Padova.

Workshop
"Origami and Mathematical thinking"

Materials:

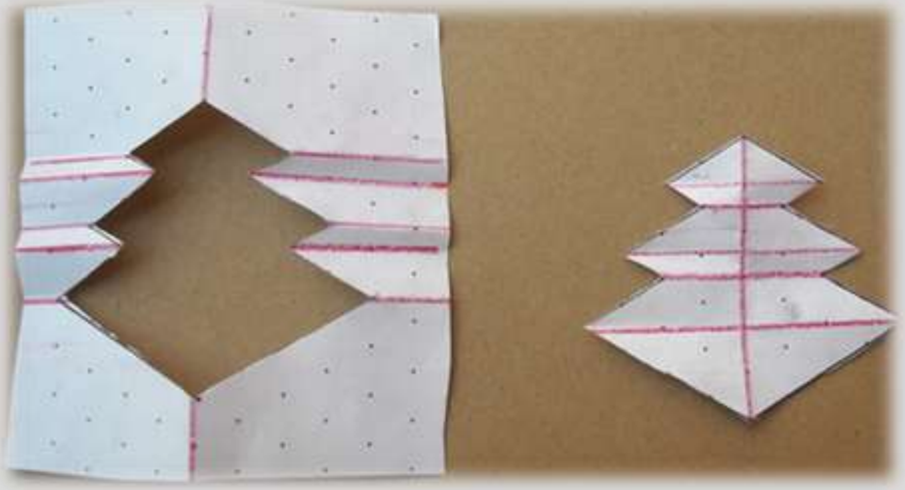
- Two A4 papers (better white, at least on one side)
- Sharp scissors

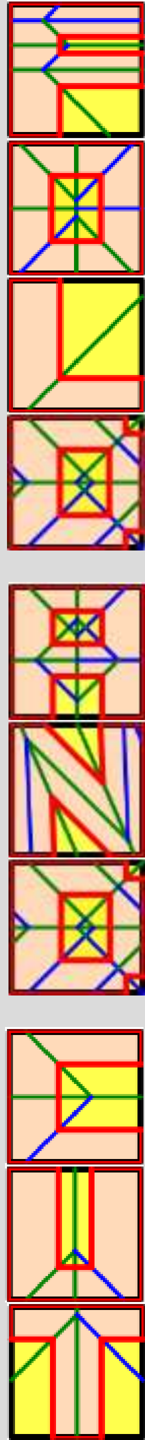


The Fold-and-Cut Problem

Fold-and-Cut process:

1. Take a piece of paper.
2. Fold it flat.
3. Make **one complete straight** cut.
4. Unfold the pieces.

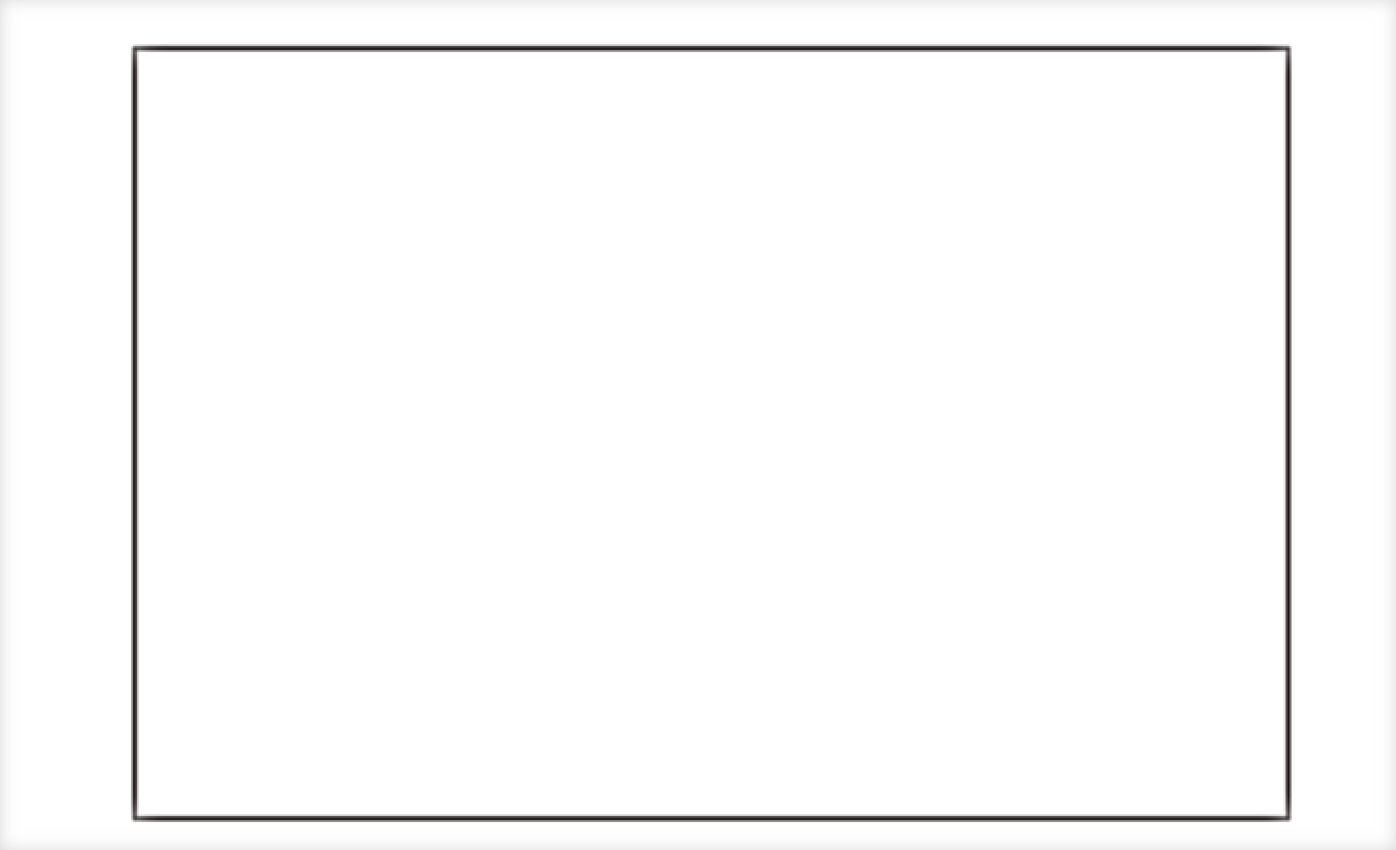




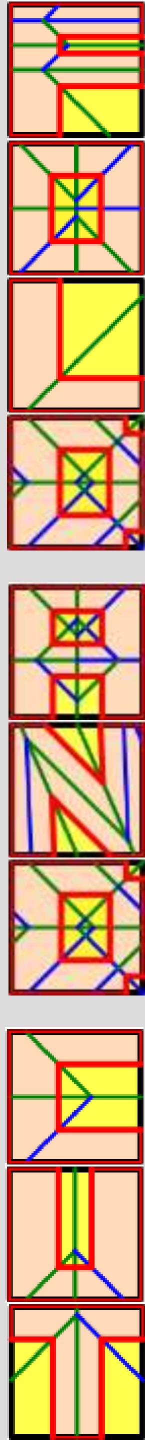
→ What shapes can result from this process?

→ Are all shapes possible?

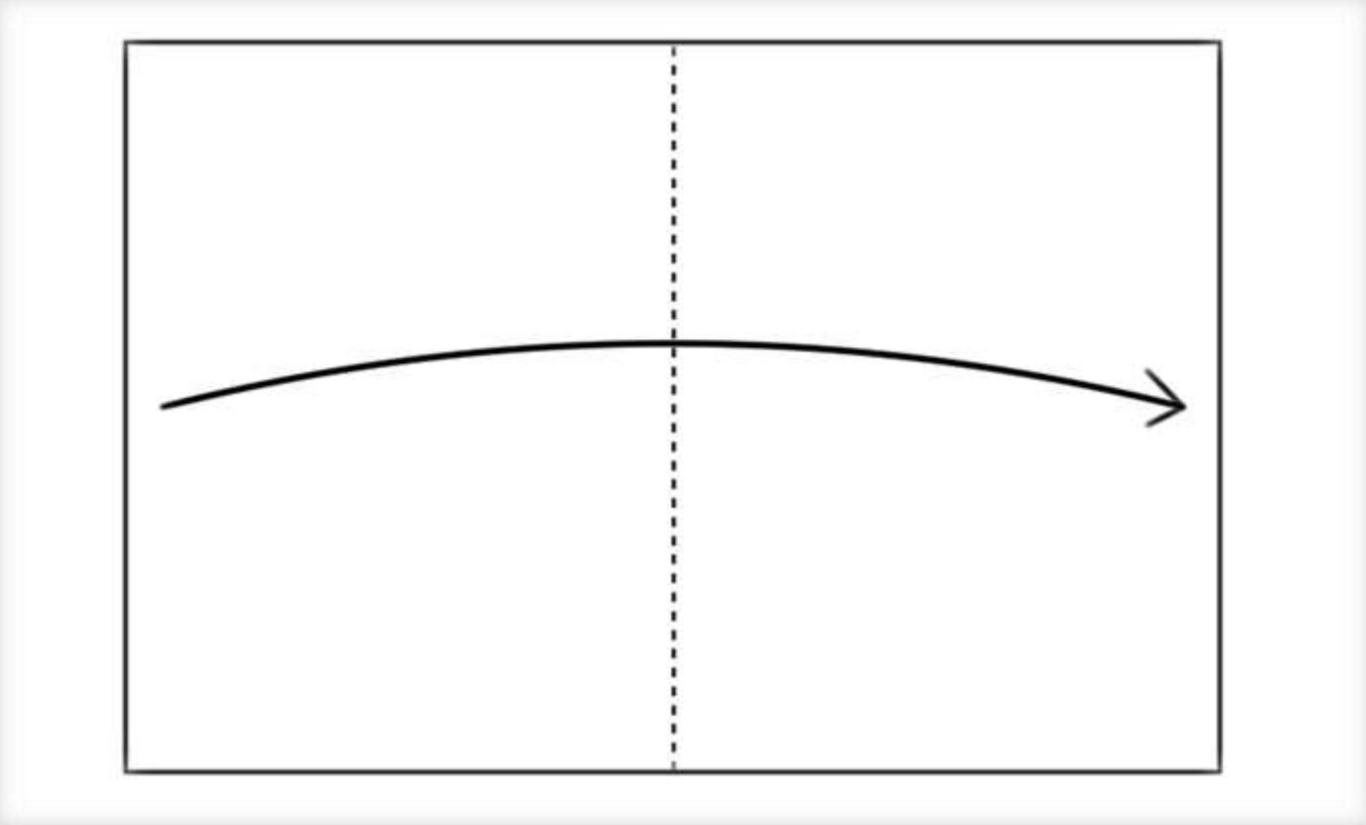
An example …



1. Take a piece of paper (for example, a rectangular piece)

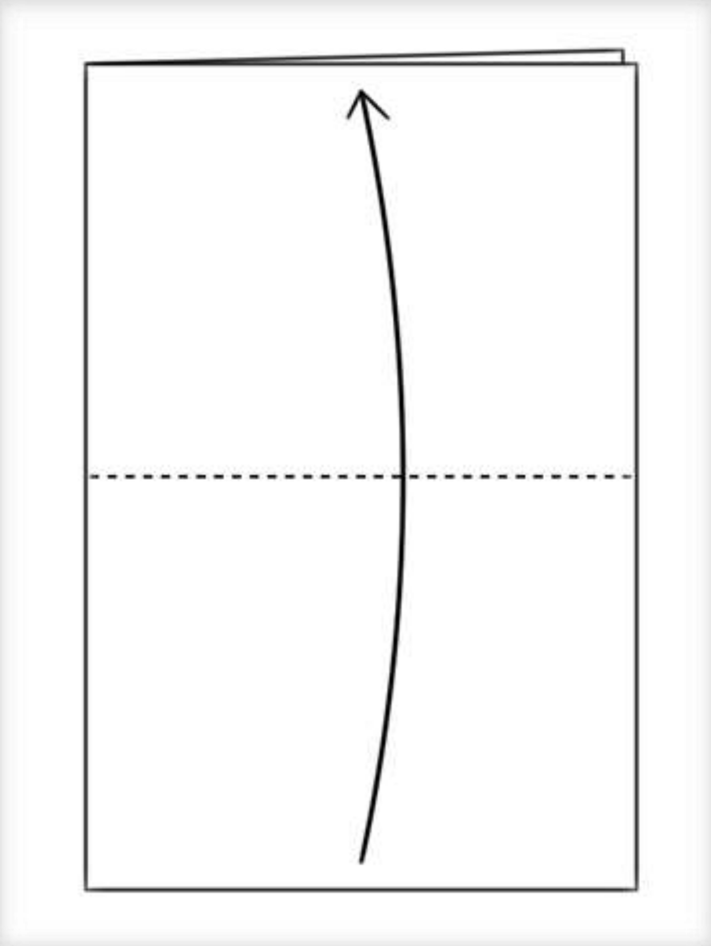


An example …

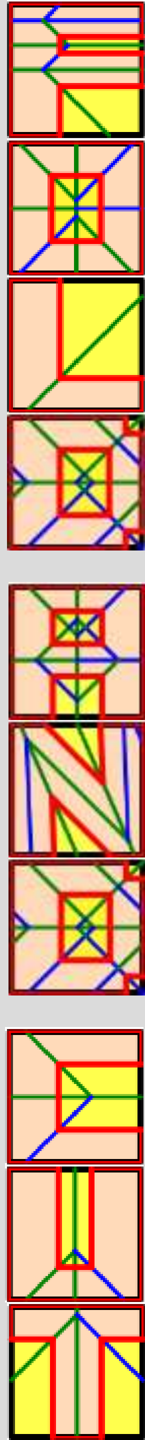


2. Fold it in half

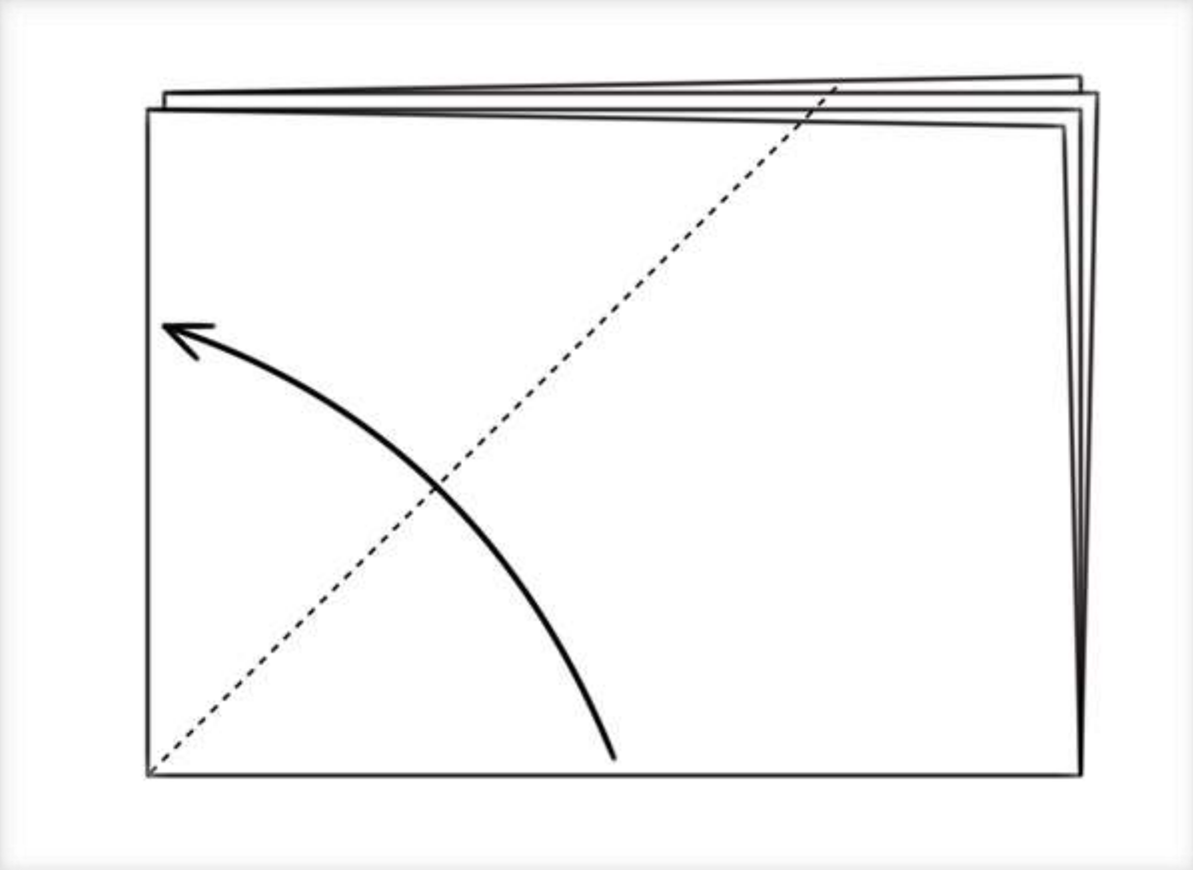
An example …



3. Fold it in half again

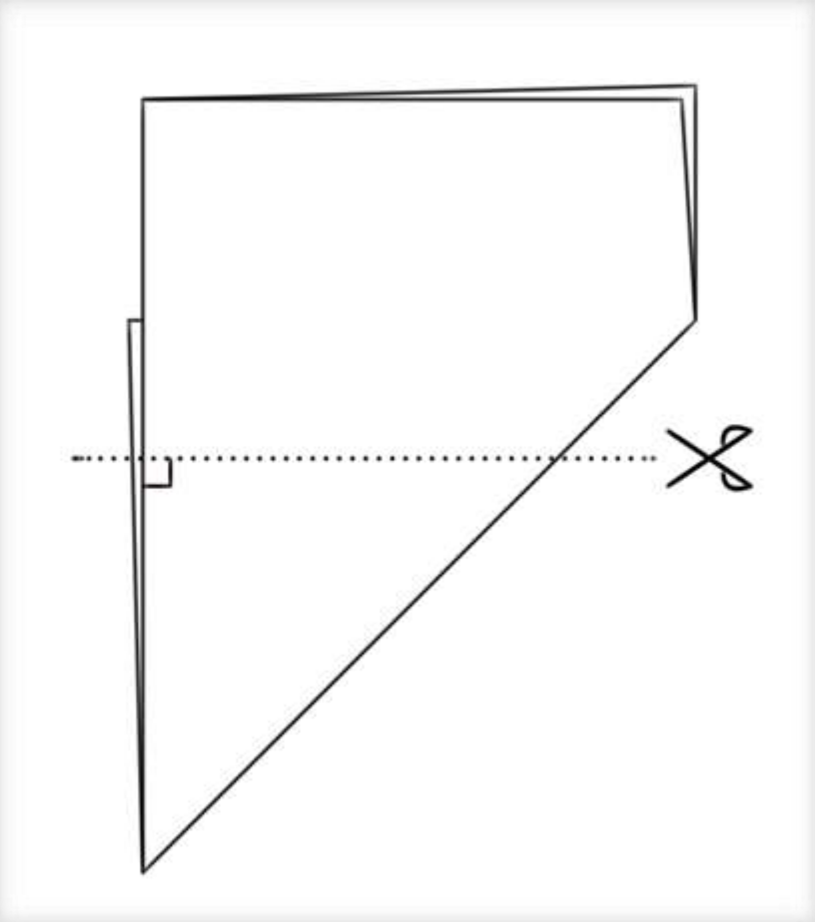


An example …

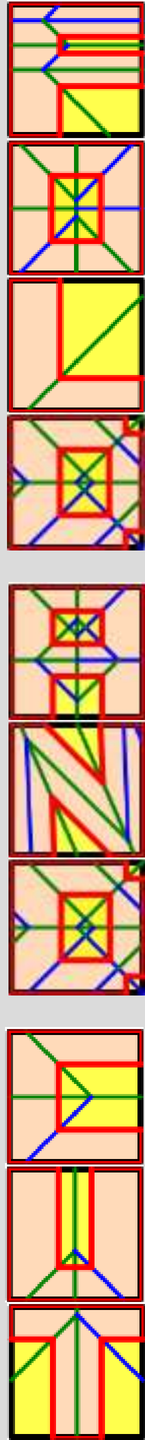


4. Fold the angle bisector of the closed angle

An example …

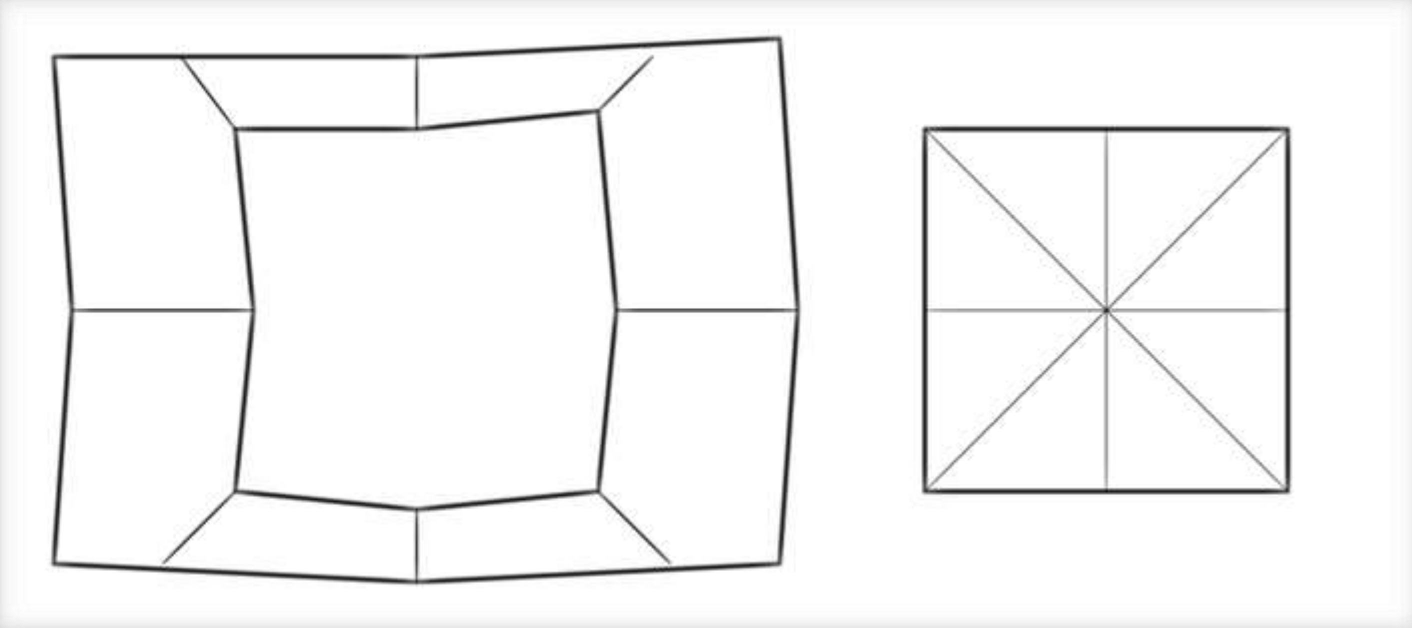


5. Cut perpendicularly to one of the sides adjacent to the closed angle



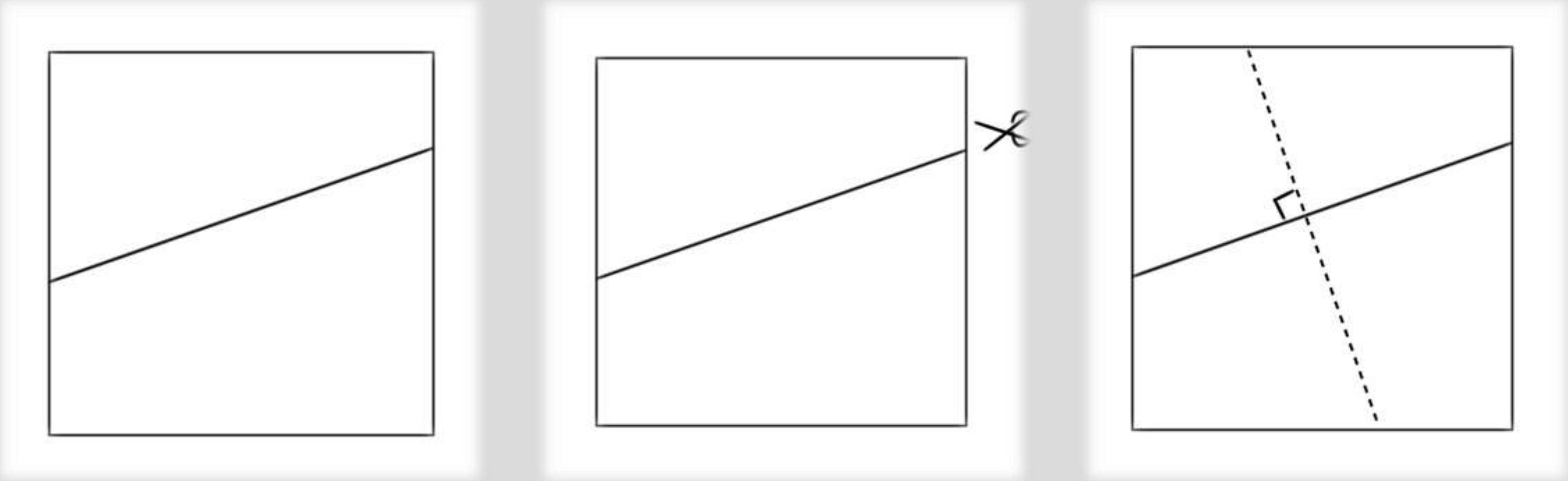
An example …

What do we get?



→ Is there a method to obtain any configuration of straight lines through the fold-and-cut process?

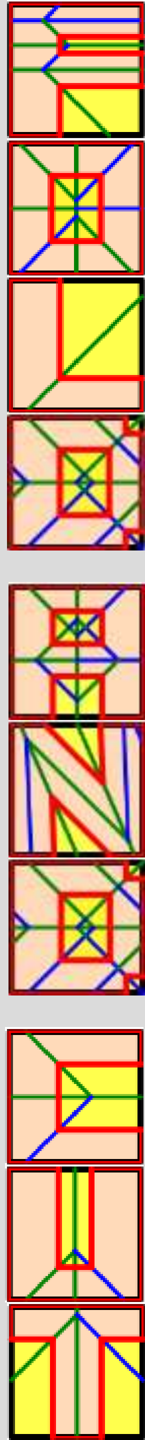
One line



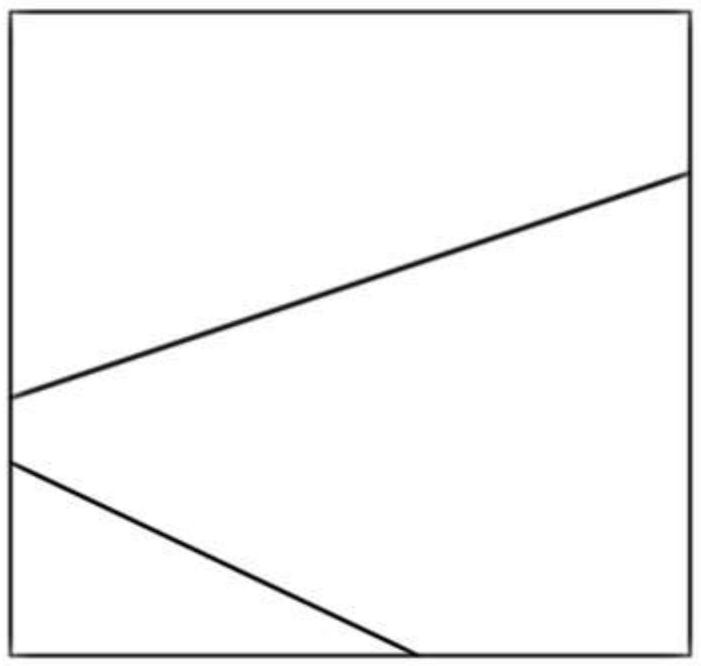
No folds

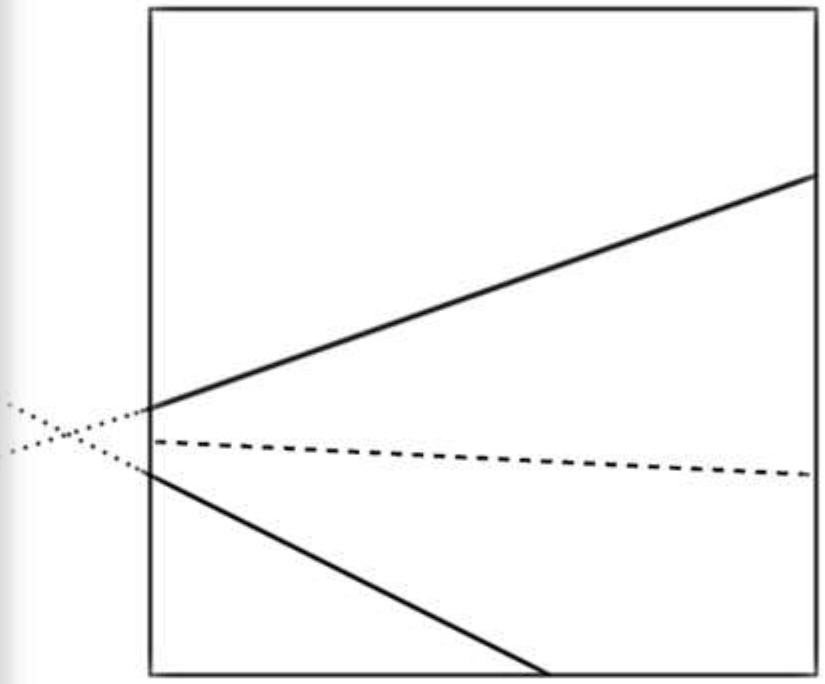
or

Folds perpendicular to the line

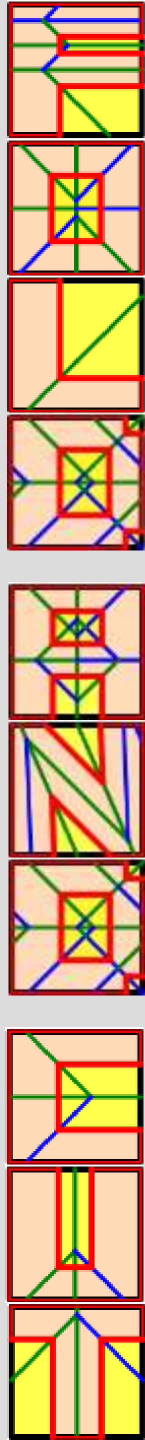


Two lines



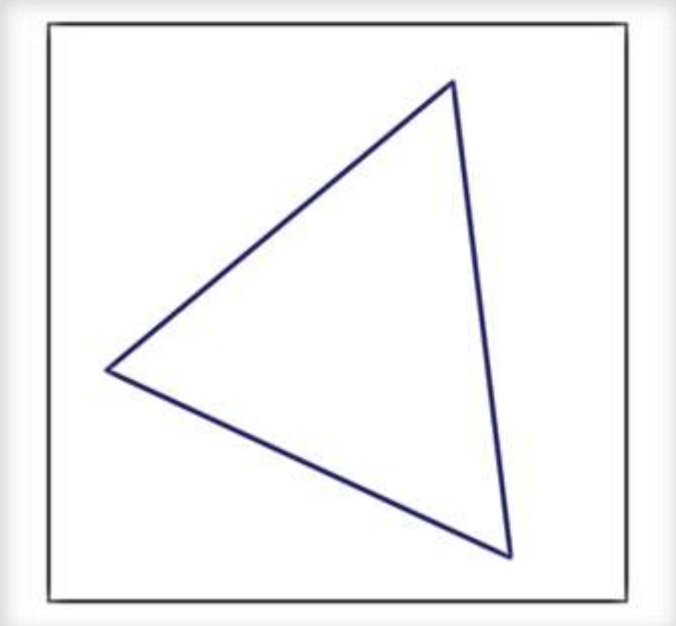


Angle bisector

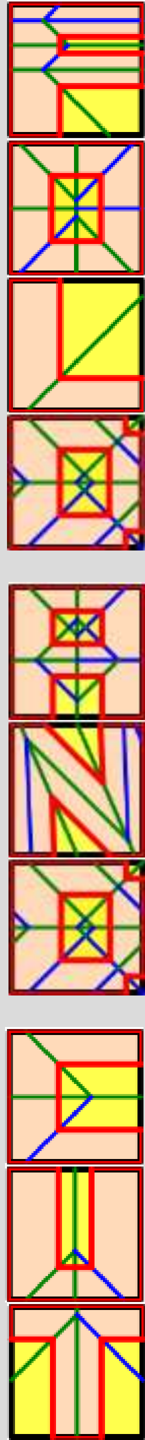


A triangle

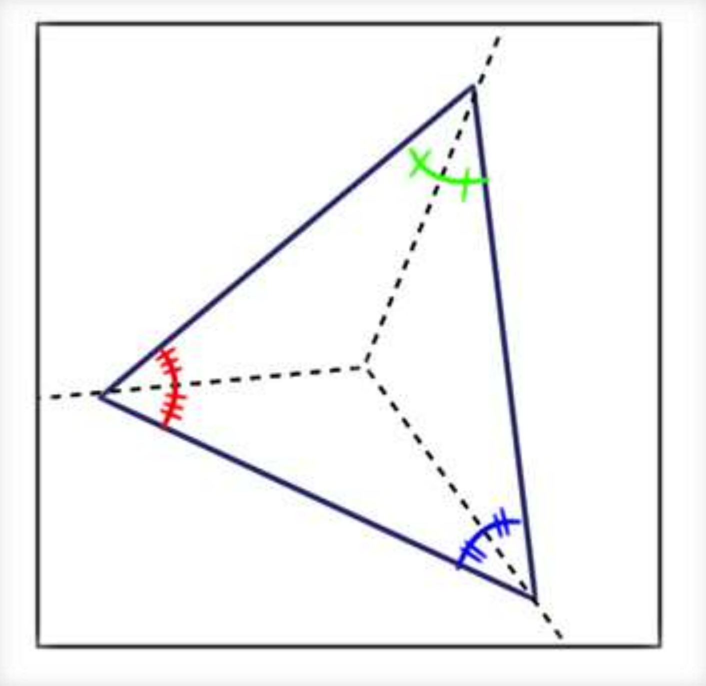
1. Draw a triangle on a piece of paper



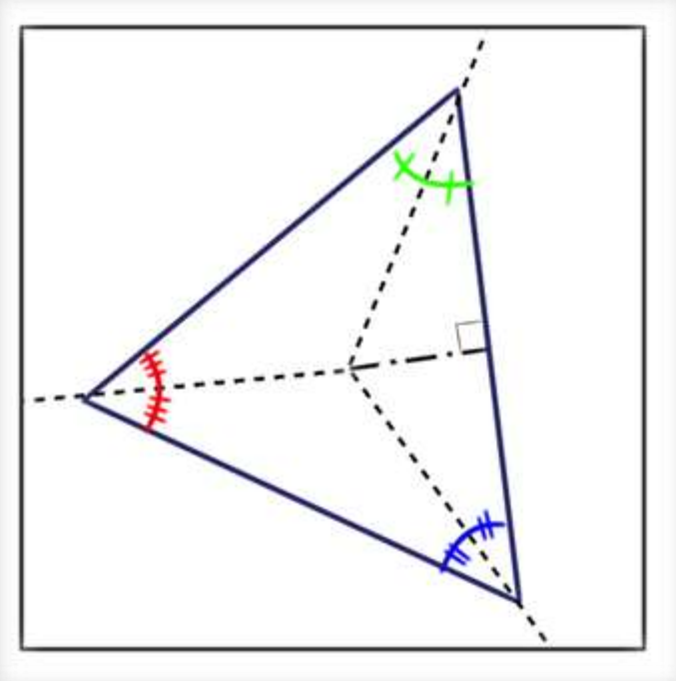
2. Find a way to fold the piece of paper in order to cut along the perimeter of the triangle with only one straight cut



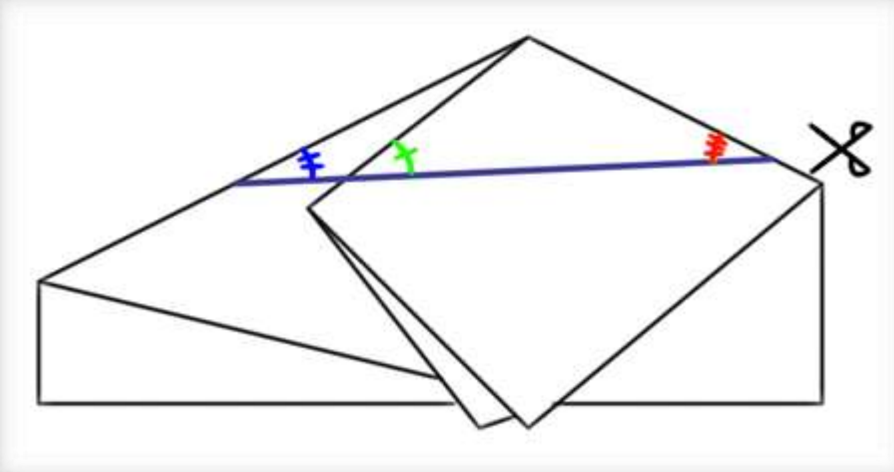
A triangle



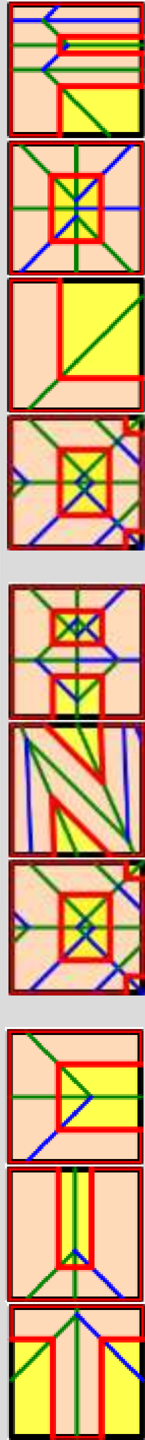
1. Fold the three bisectors



2. Flatten the model



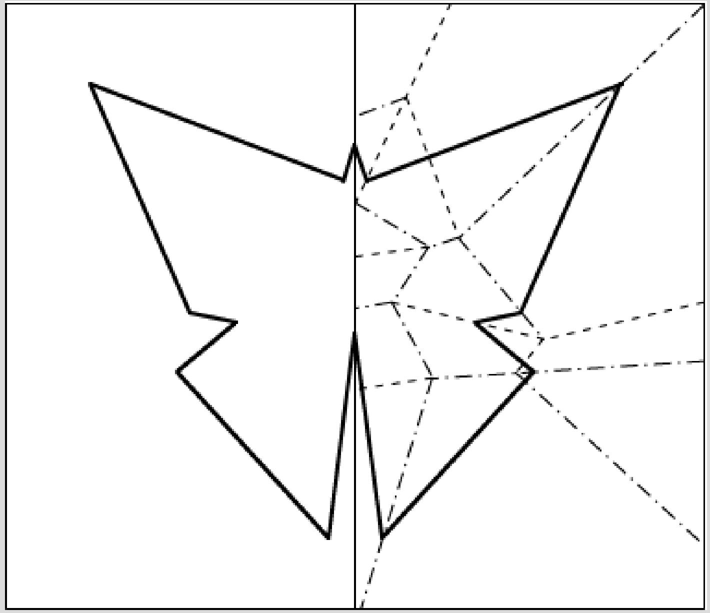
3. Cut

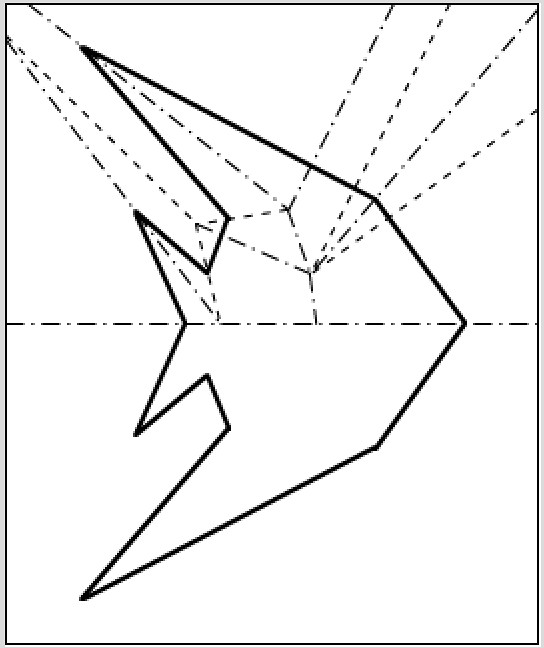


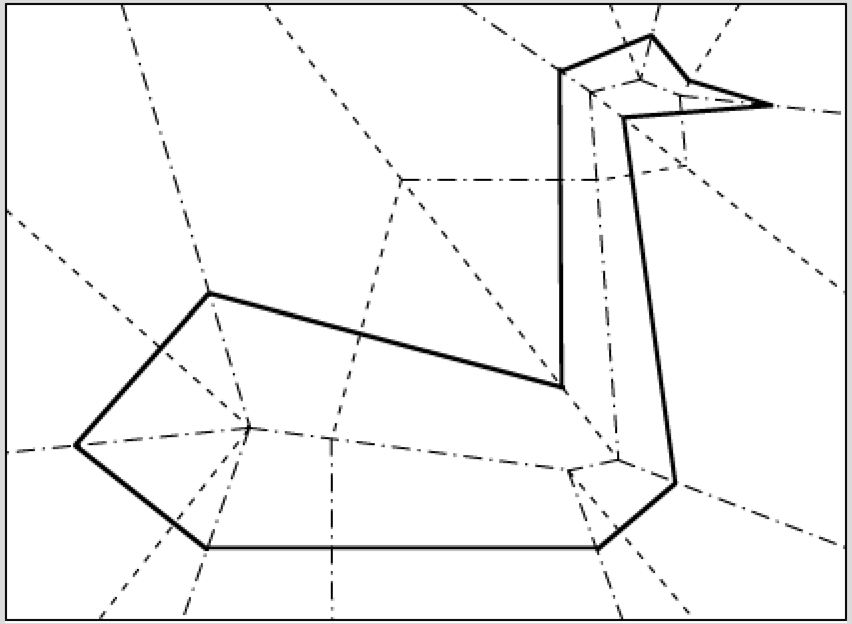
The Fold-and-Cut Theorem

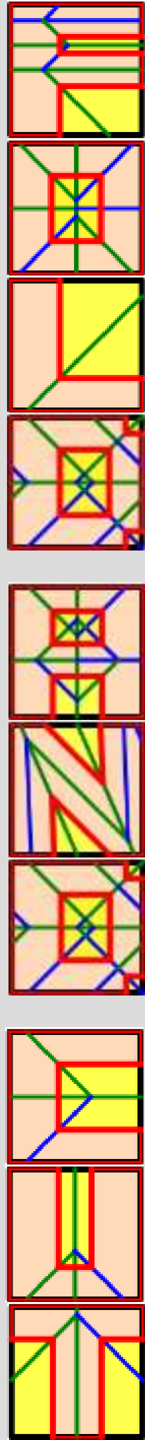
Every pattern (plane graph) of straight-line cuts can be made by folding flat and one complete straight cut.

(Demaine, Demaine & Lubiw 1998)



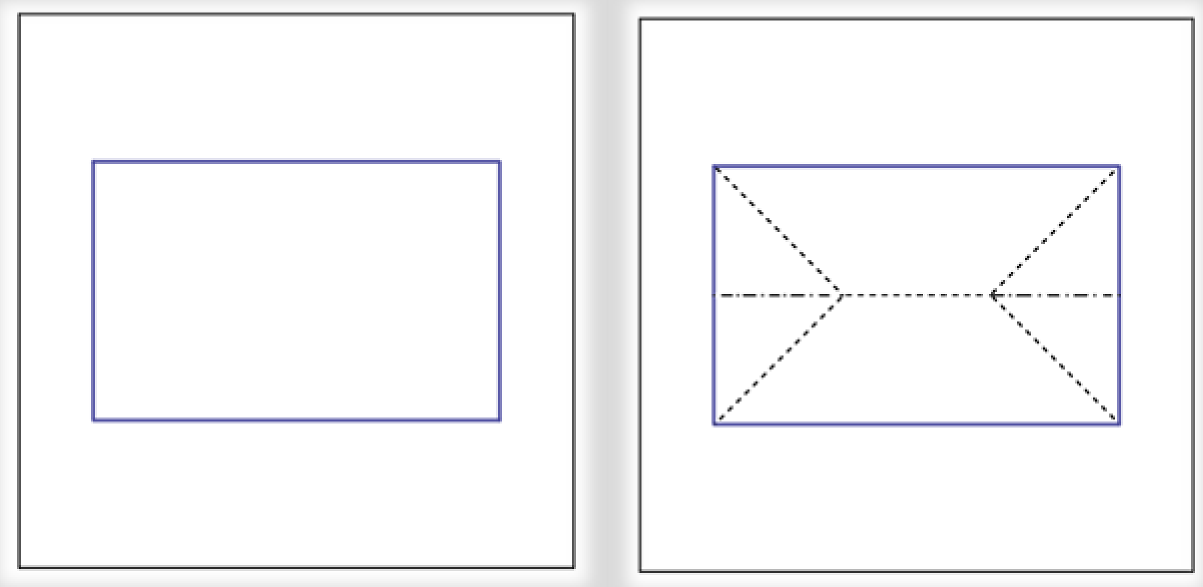






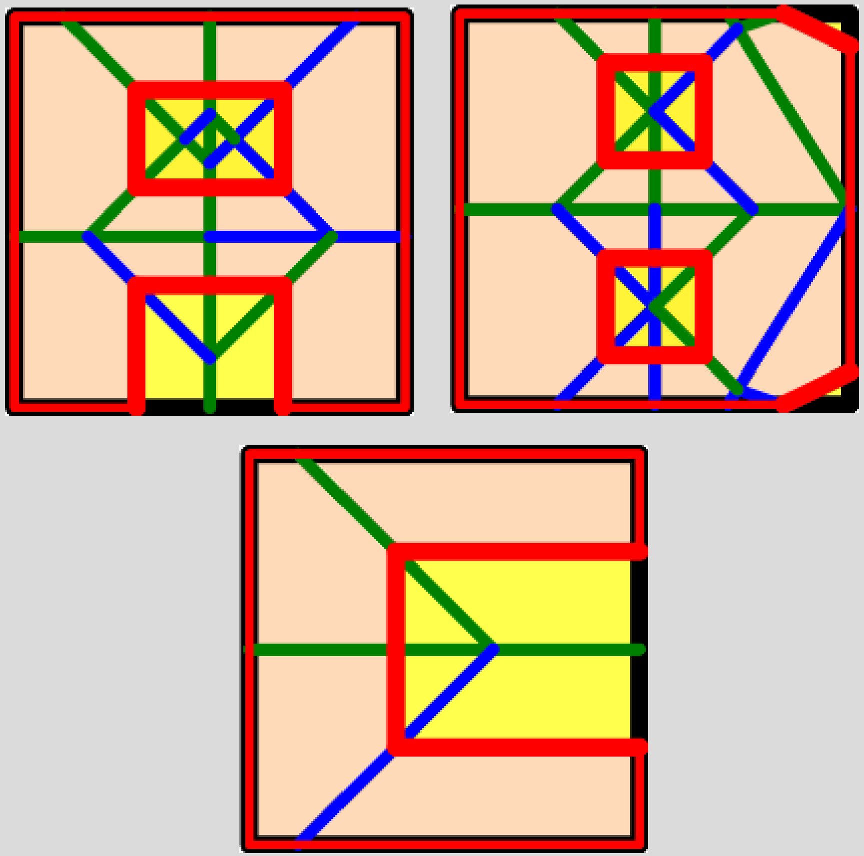
Other examples

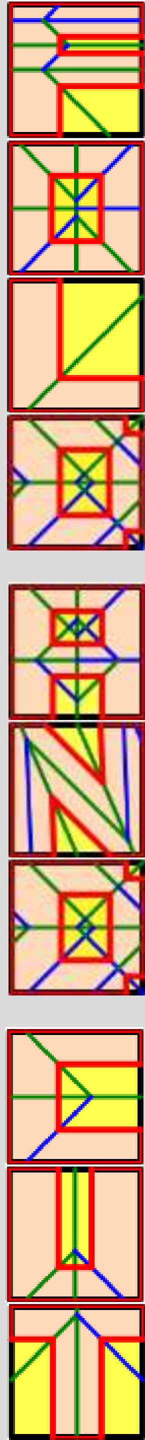
RECTANGLE



ALPHABET

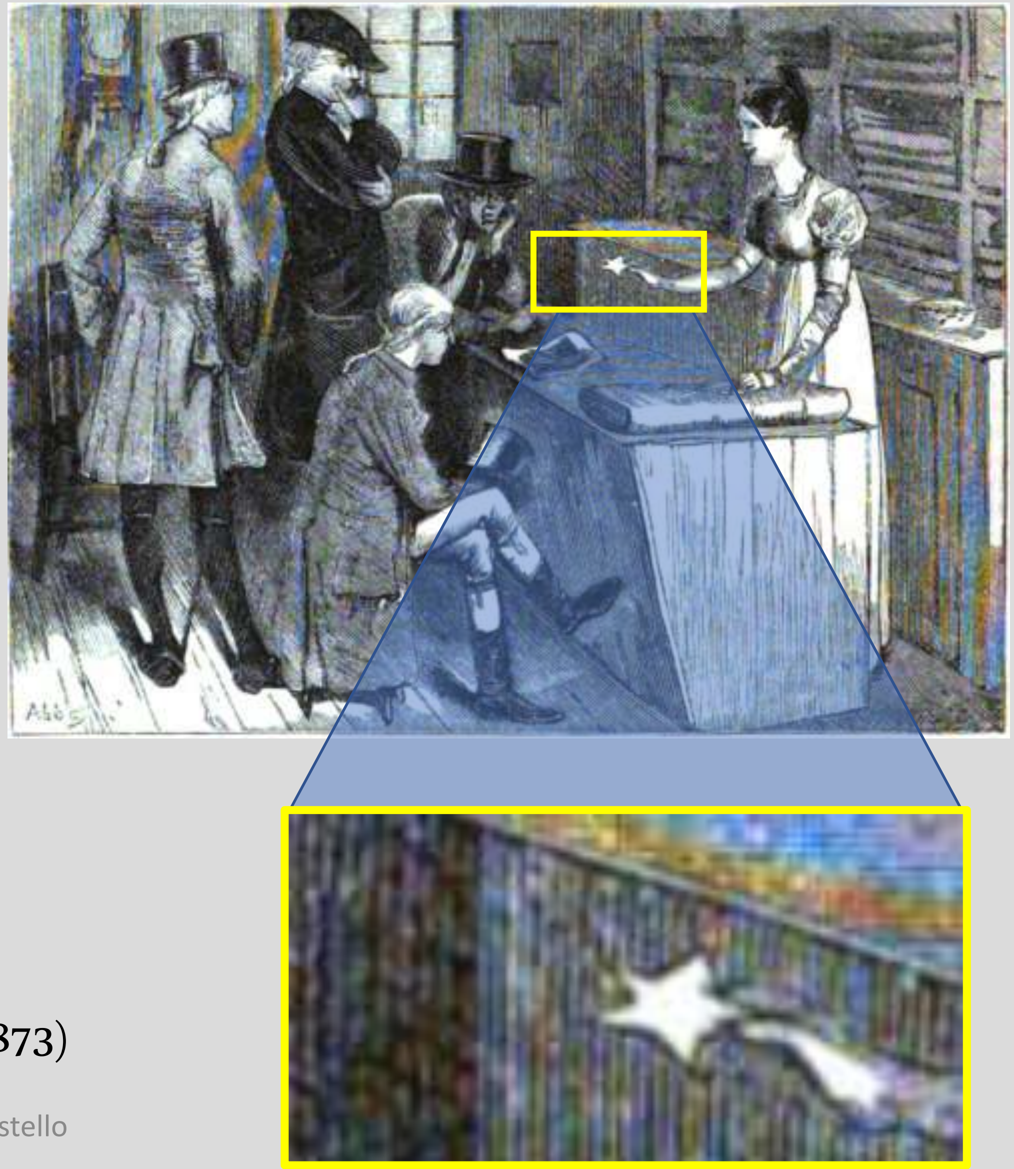
LETTERS



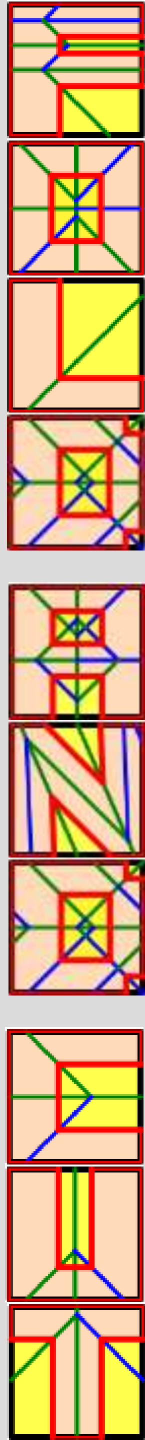


The Betsy Ross Star

In 1777 George Washington and a committee of the Congress showed Betsy Ross planes for a flag with thirteen six-pointed stars, and asked her whether she could make such a flag. She said that she would be willing to try, but suggested that the stars should have five points. To support her argument, she demonstrated how easily a regular five-pointed star could be made by folding a piece of paper flat and making one cut with scissors.

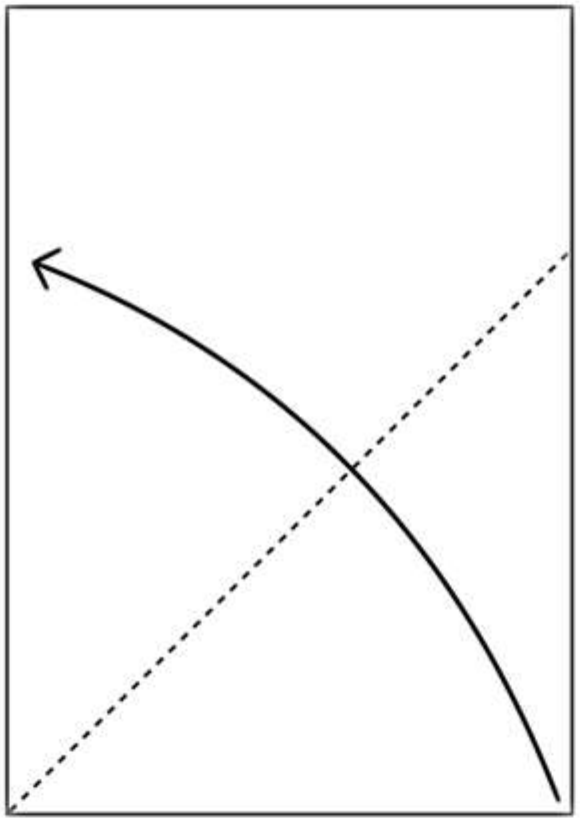


(National standards and emblems. *Harper's New Monthly Magazine*, 1873)

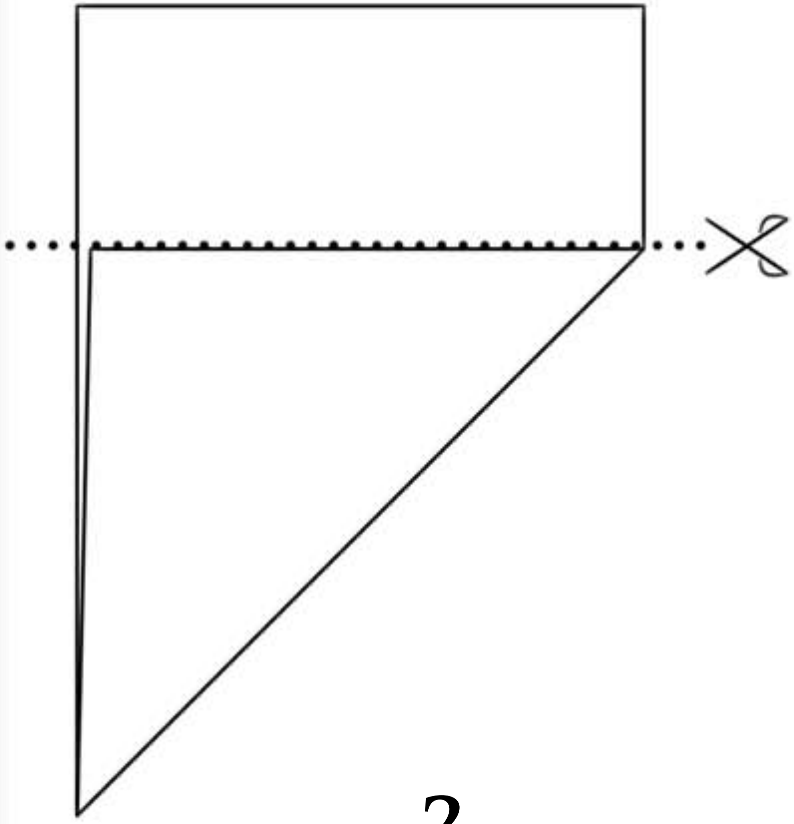


The Betsy Ross Star

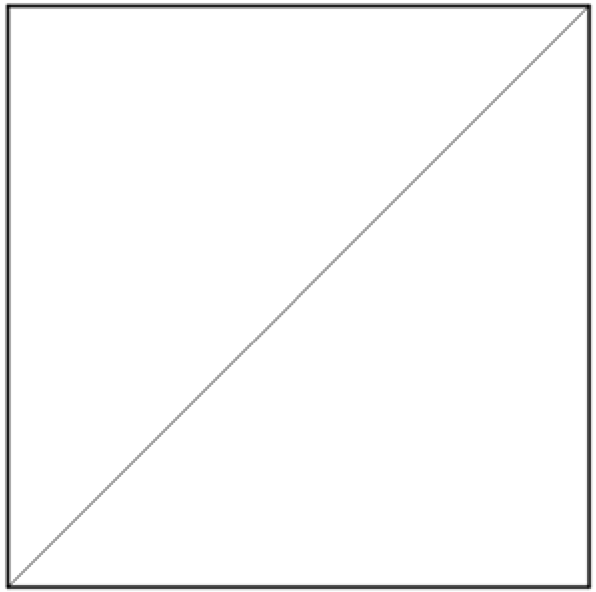
Prepare a squared piece of paper:



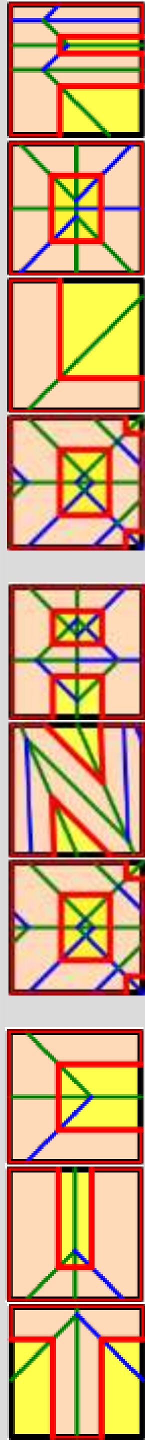
1.



2.

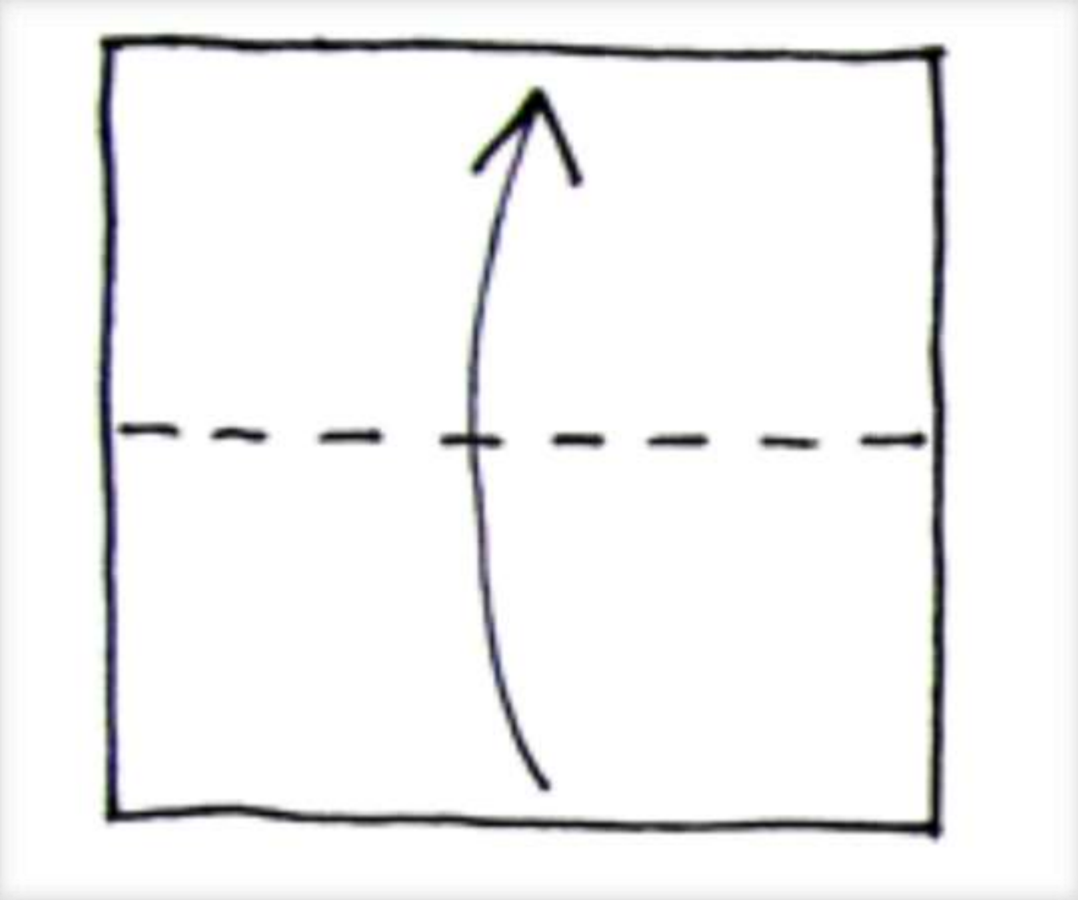


3.

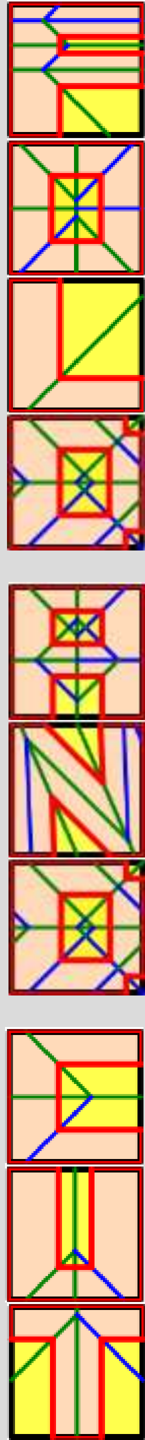


The Betsy Ross Star

(Diagrams by Mary Ellen Palmeri)

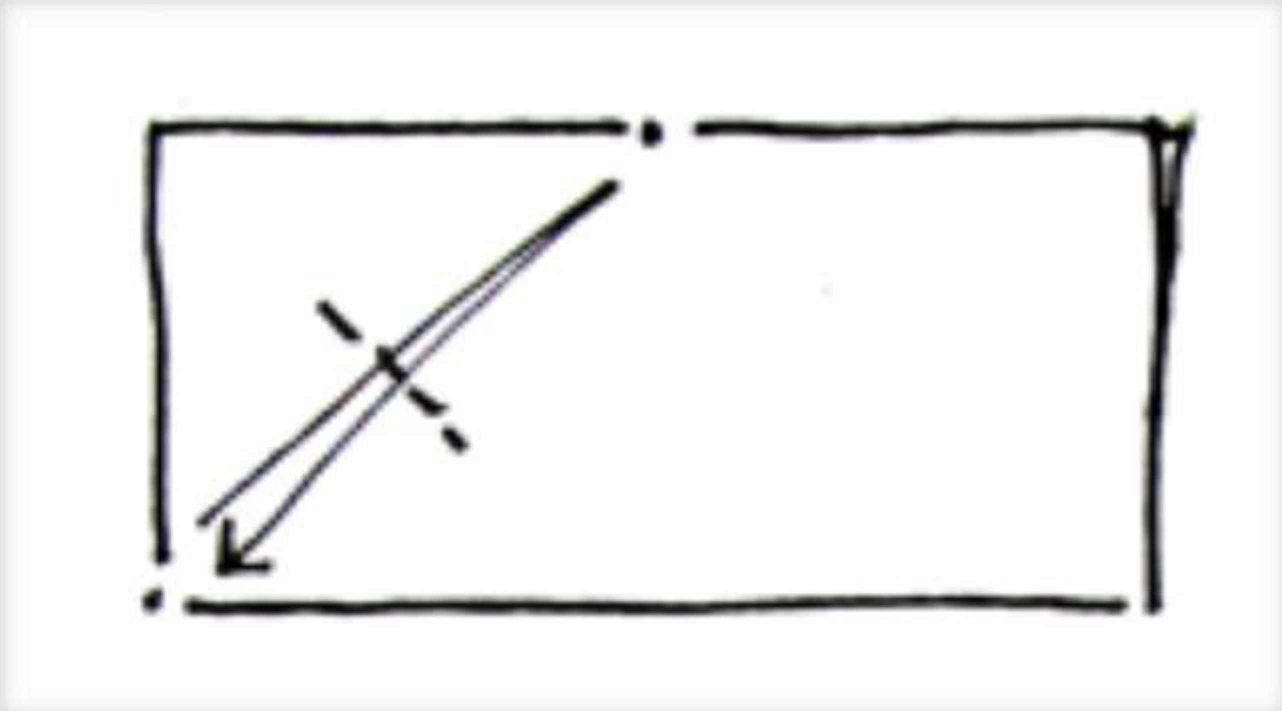


1. Fold a square in half (the bottom to the top)

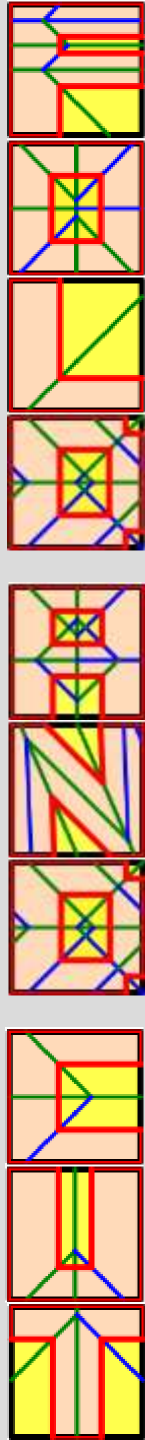


The Betsy Ross Star

(Diagrams by Mary Ellen Palmeri)

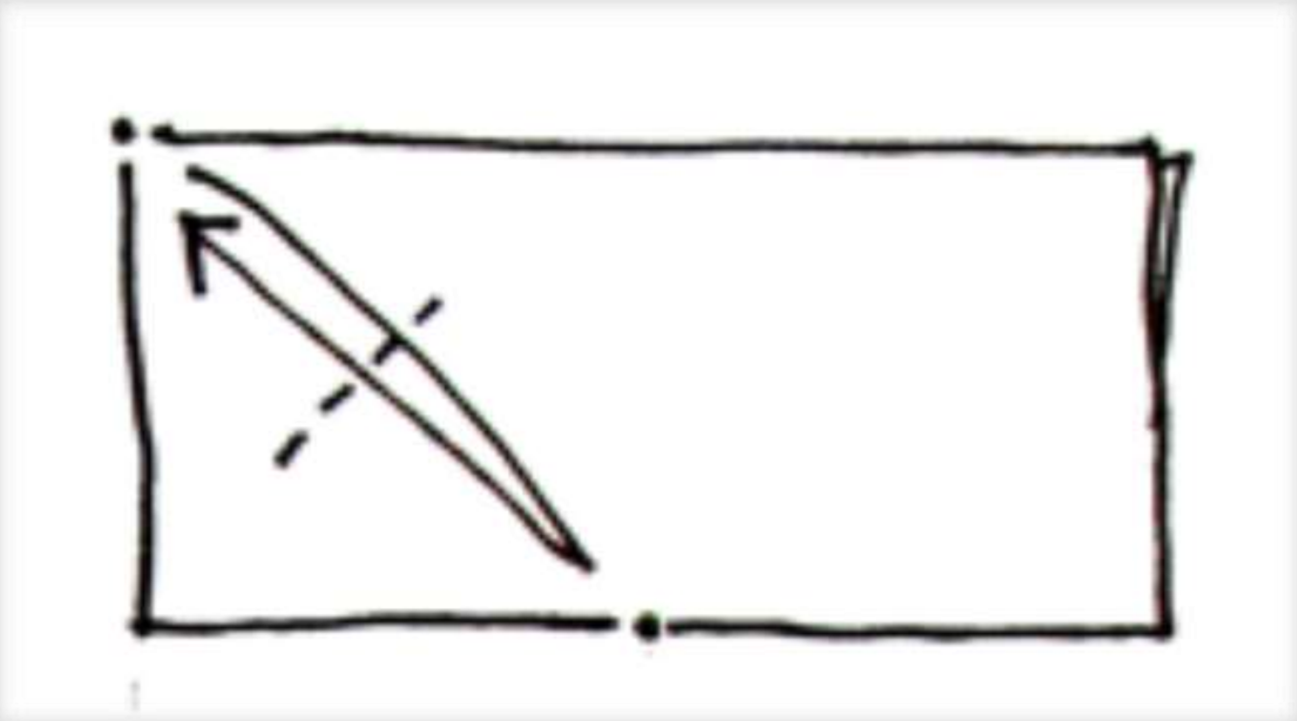


2. Softly fold the bottom left side up to the top edge; press only the center of the fold to make a small crease mark, then unfold

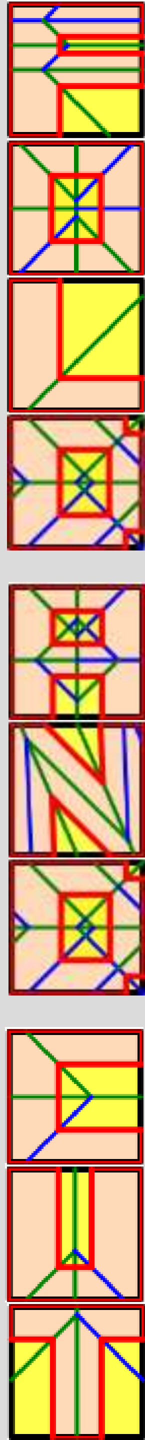


The Betsy Ross Star

(Diagrams by Mary Ellen Palmeri)

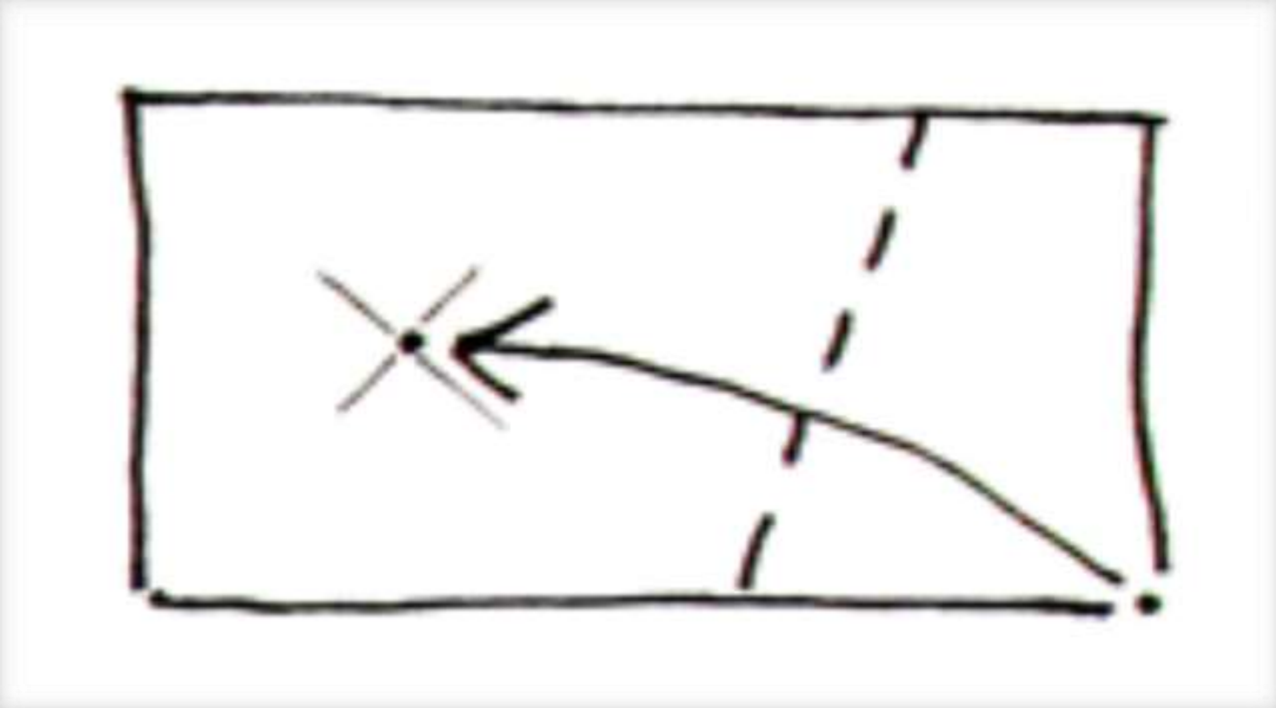


3. Bring the upper left corner to the bottom edge, aligning the vertical side with the bottom; press to make a mark crossing the previous one, then unfold

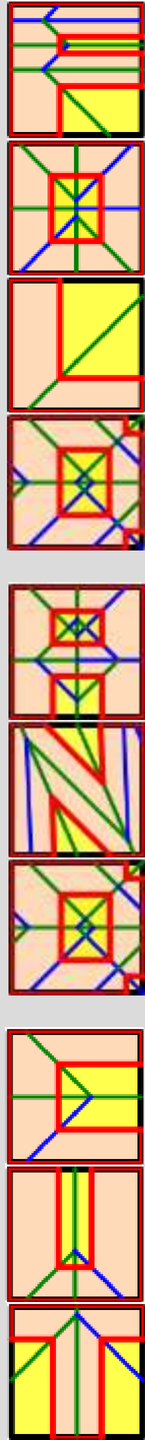


The Betsy Ross Star

(Diagrams by Mary Ellen Palmeri)

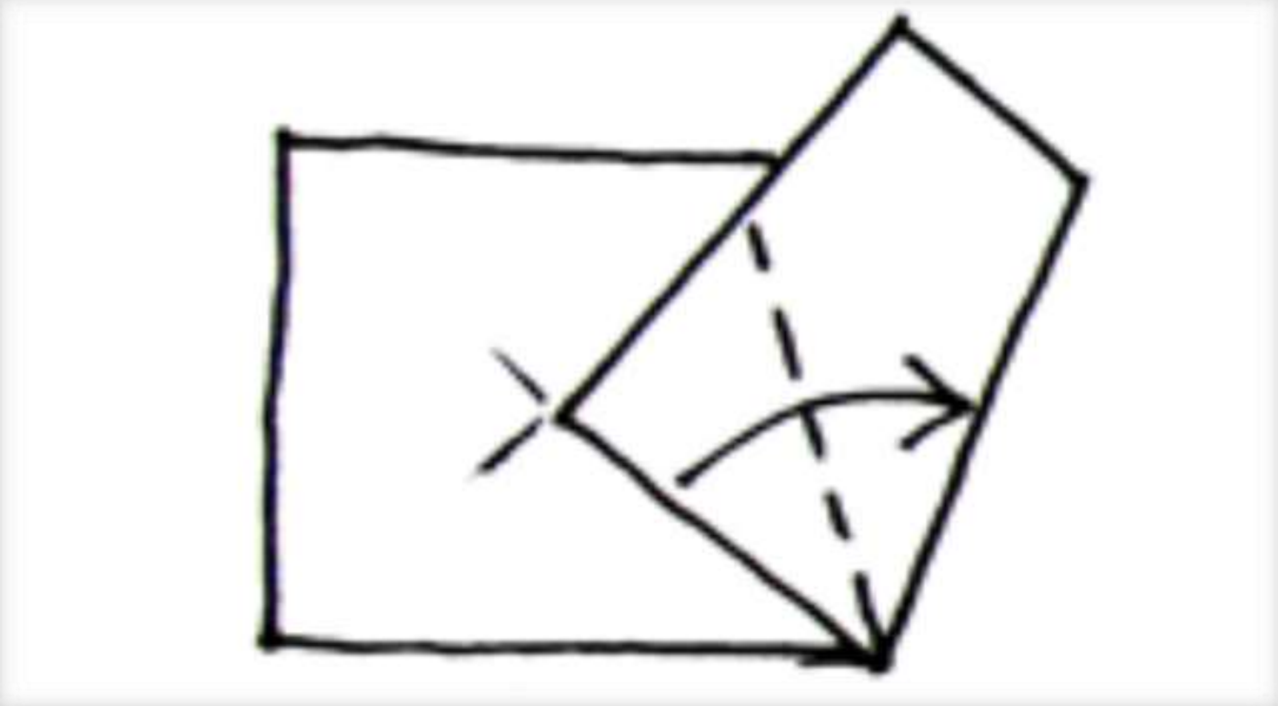


4. You have creased an X; bring the lower corner to the center of the X and firmly crease the paper

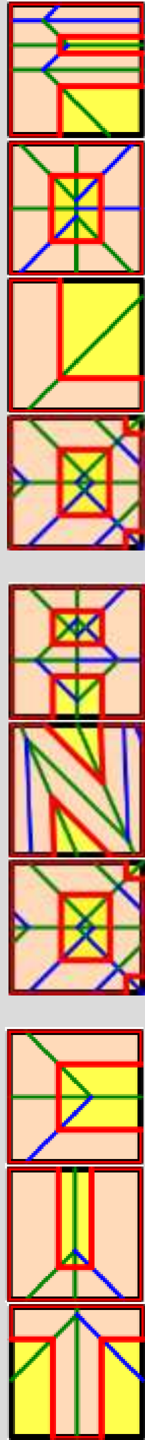


The Betsy Ross Star

(Diagrams by Mary Ellen Palmeri)

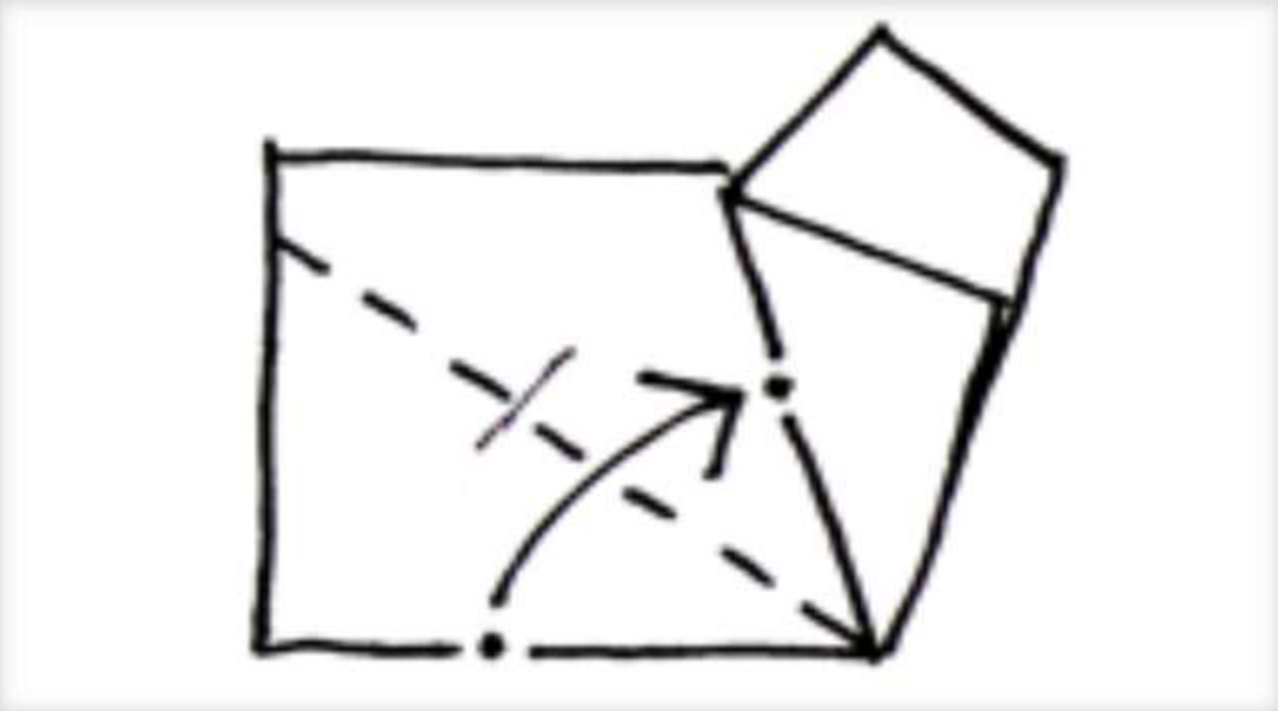


5. Fold back the edge of this flap to make it lie exactly on the right side of the model

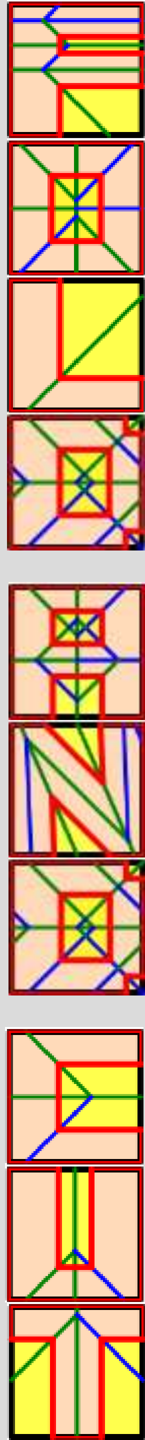


The Betsy Ross Star

(Diagrams by Mary Ellen Palmeri)

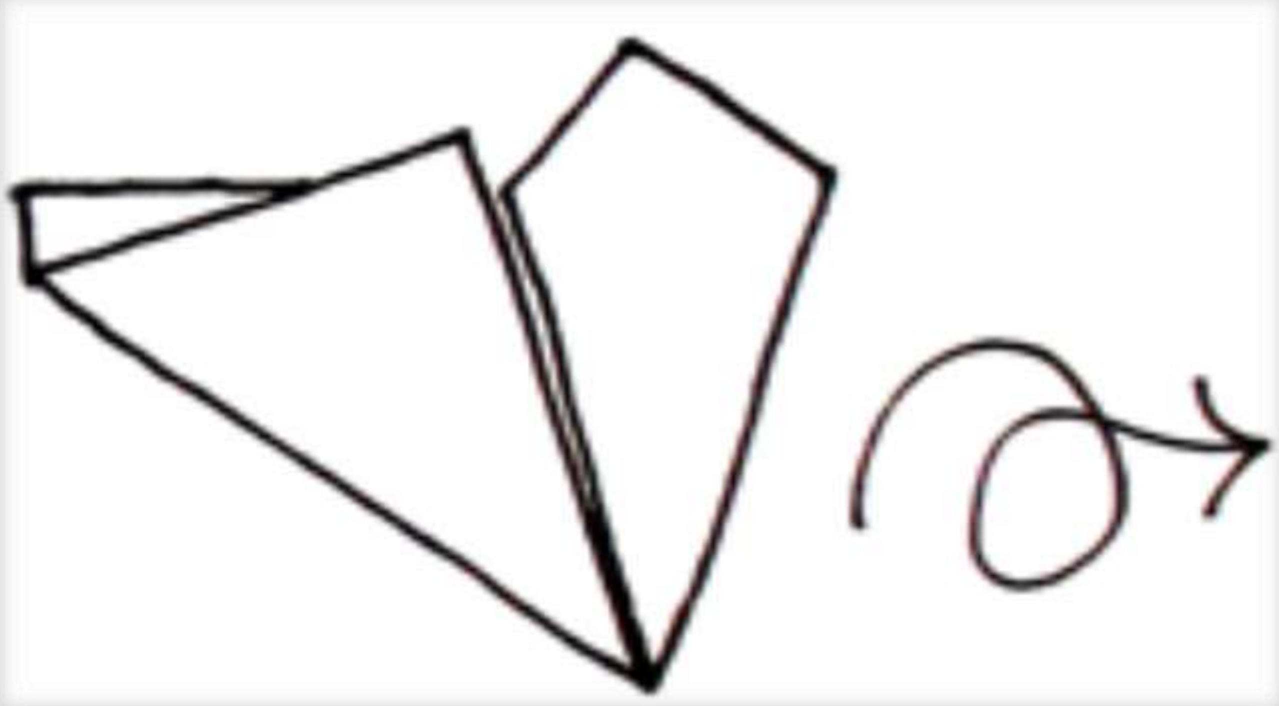


6. Fold the bottom left side up to coincide with the inside edge of the flap

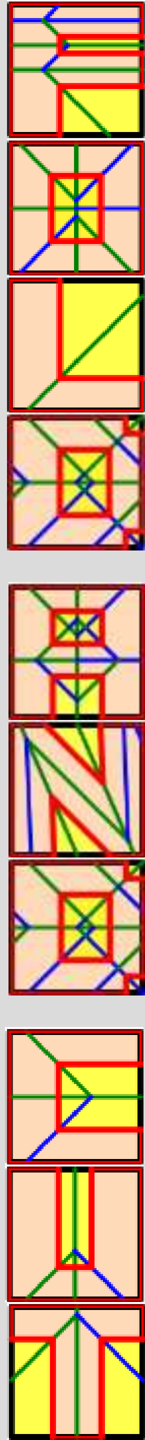


The Betsy Ross Star

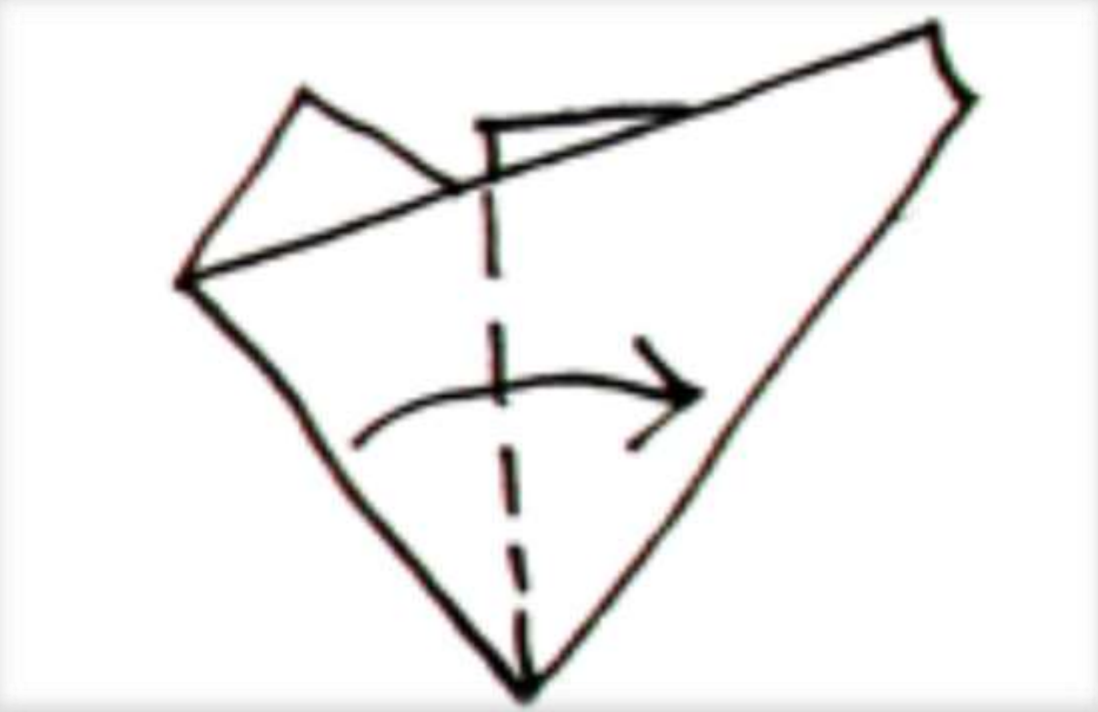
(Diagrams by Mary Ellen Palmeri)



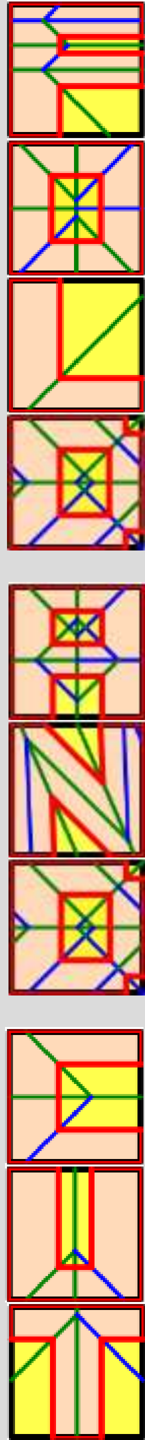
7. Flip the model



The Betsy Ross Star
(Diagrams by Mary Ellen Palmeri)

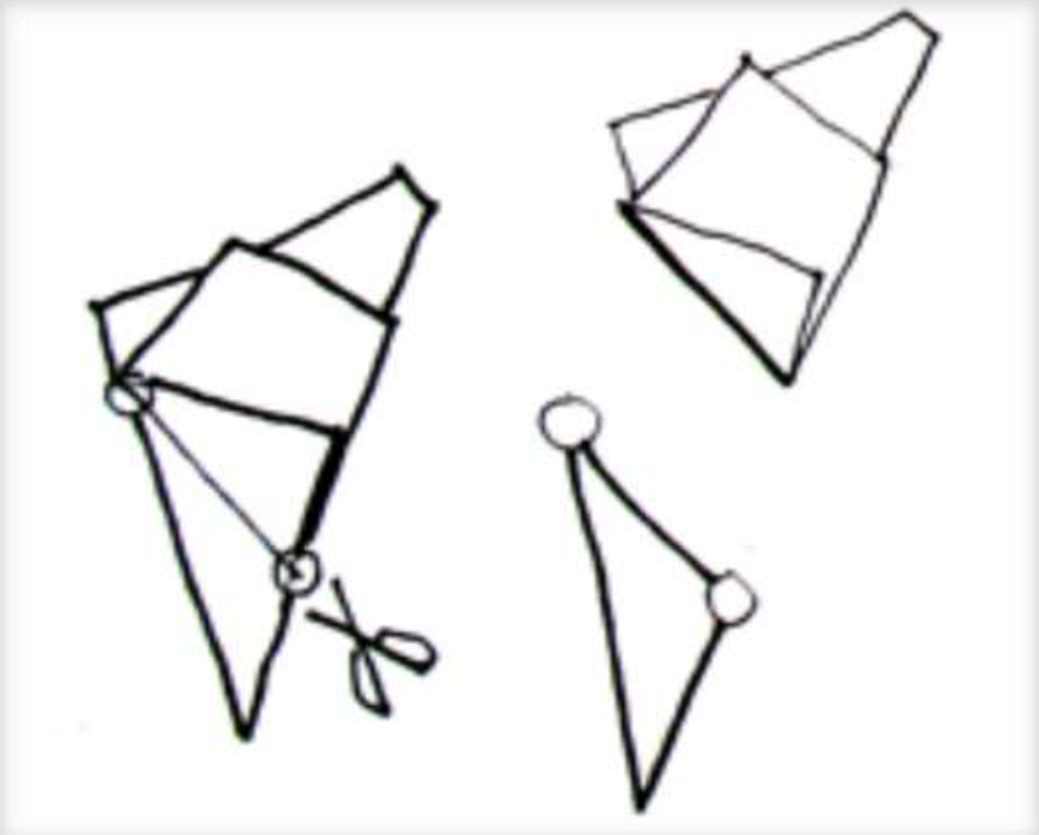


8. Fold the left edge to coincide with the right edge, folding the V shape in half

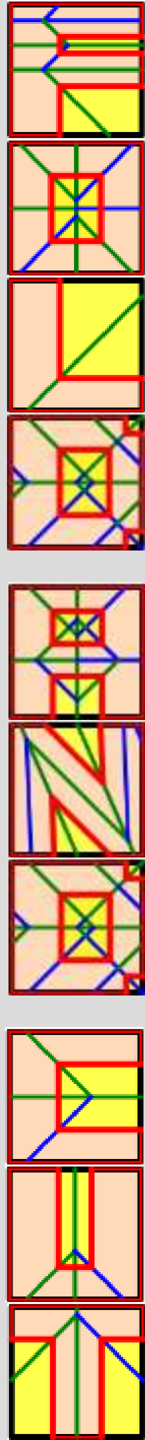


The Betsy Ross Star

(Diagrams by Mary Ellen Palmeri)

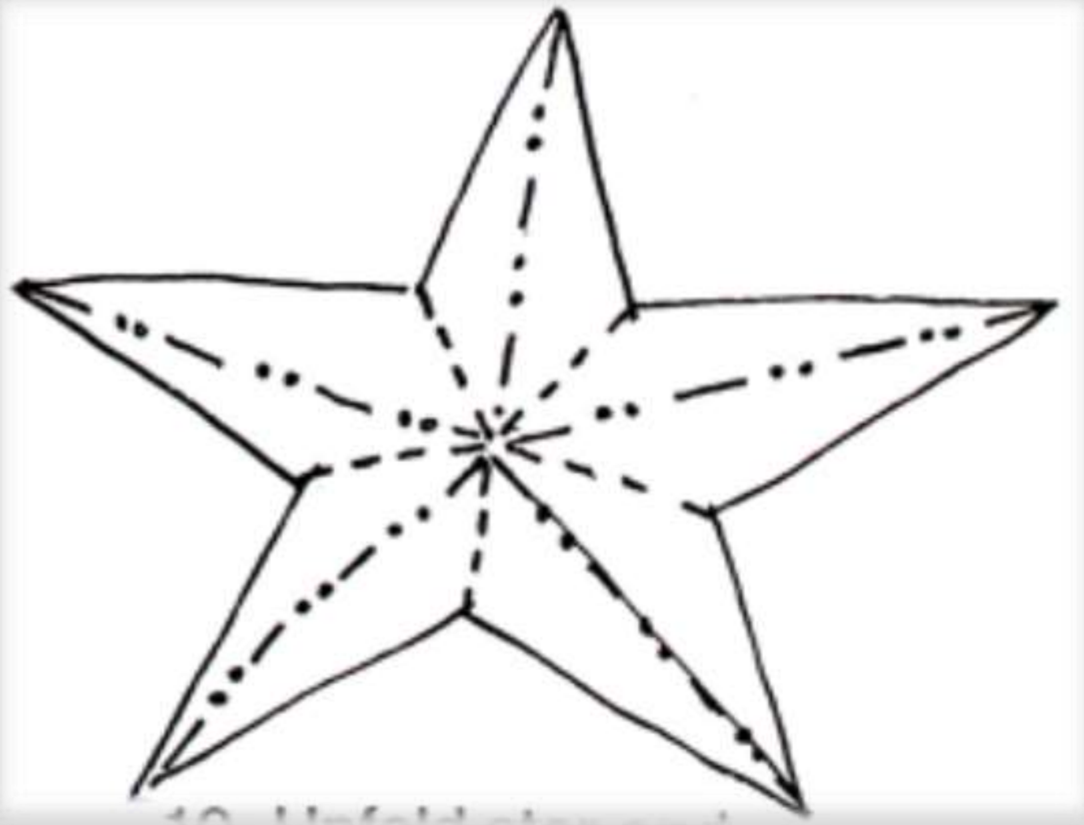


9. Cut all layers at a sharp angle as shown, from the middle point on the right edge to the highest left point of the same flap

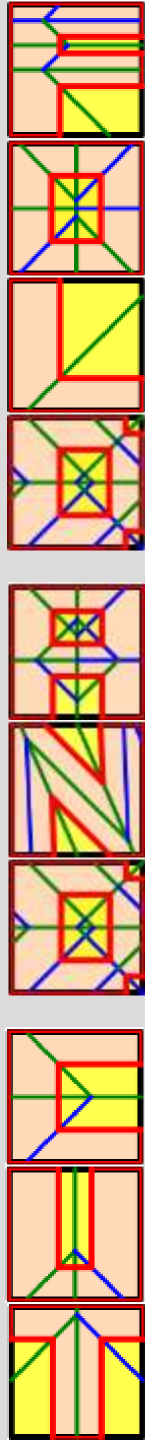


The Betsy Ross Star

(Diagrams by Mary Ellen Palmeri)



10. Unfold, then change to mountain the folds from the center to the points of the star, and to valley the folds in the other cases

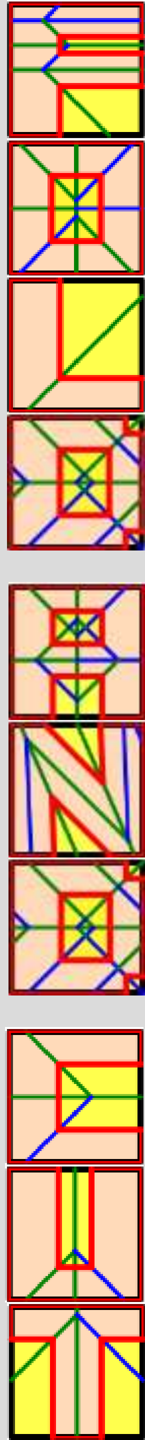


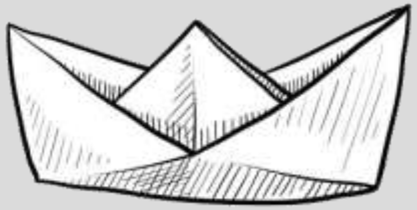
Bibliography

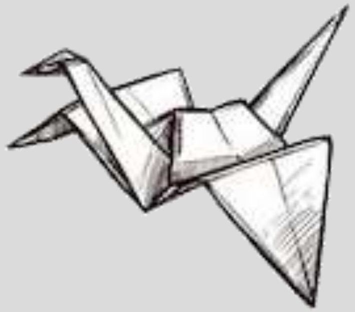
- Demaine, E. D., Demaine, M. L. & Lubiw A. (1998). Folding and cutting paper. In *Proc. Japan Conf. Discrete and Computational Geometry*, Tokyo.
- National standards and emblems. *Harper's New Monthly Magazine*, 47(278): 171-181, 1873 (https://babel.hathitrust.org/cgi/pt?id=uc1.31175021862241&view=1up&seq=193&size=125)

Links and material

- "Fold & one cut" Lecture by Erik Demaine at MIT OpenCourseWare: https://www.youtube.com/watch?v=K0GuKDSX1FA
- One-cut alphabet and numbers: https://erikdemaine.org/fonts/simplefoldcut/
- One-cut examples: https://erikdemaine.org/foldcut/examples/







Thank you