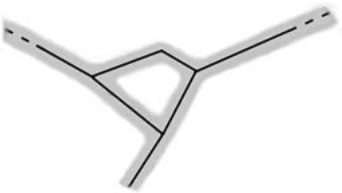
Nonlinear Schrödinger dynamics on metric graphs
(A variational approach)

Alice Ruighi
(Adami R., Boni F., Dovetta S., Serra E., Tentarelli L., Tilli P.)

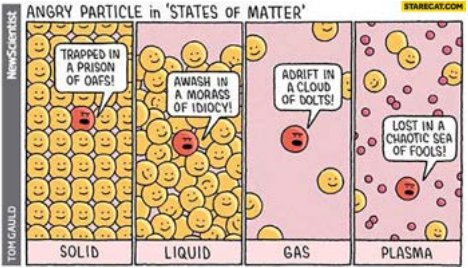
Dipartimento di Scienze Matematiche "G.L. Lagrange"
Politecnico di Torino

Nonlinear Schrödinger dynamics on metric graphs..Why??

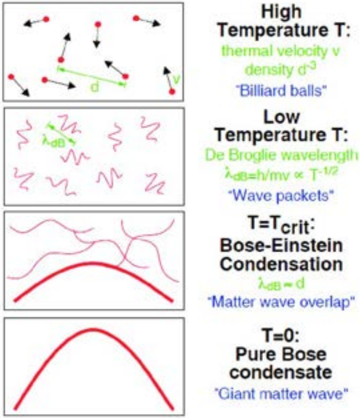
- Graphs provide one-dimensional approximations for constrained dynamics in which transverse dimensions are negligible compared to longitudinal ones.
- Bose-Einstein condensates (BEC).



Bose-Einstein condensates.. The fifth state of matter in a nutshell.

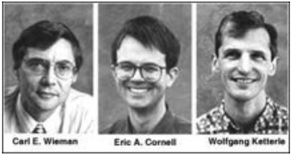


In the early '20s, **Satyendra Nath Bose** and **Albert Einstein** predicted that..



But only 70 years later, in 1995, **Wieman**, **Cornell** and **Ketterle** proved the existence of BEC experimentally!

The **Nobel Prize** in Physics 2001.



..and so what? Why "variational"?

At the **absolute zero** temperature (0° Kelvin, -273.15° Celsius, -459.67° Farenheit), all the particles of the ultracold gas of identical and indistinguishable bosons, share the same **wave function** φ that solves the **variational problem**

$$\min_{\substack{u \in H^1(\Omega), \\ \int |u|^2 = N}} E_{GP}(u).$$

Ω is the trap in which the particles are confined, N is the number of the particles of the system and finally E_{GP} is the Gross-Pitaevskii functional defined as

$$E_{GP}(u) = ||\nabla u||^2_{L^2(\Omega)} + 8\pi\alpha ||u||^4_{L^4(\Omega)},$$

where α is the scattering length of the two-body interaction between the particles in the condensate.

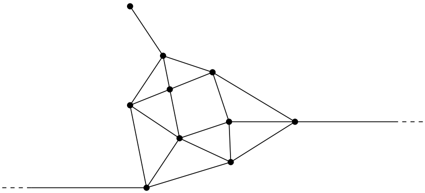
Our problem:

Defined the energy functional as

$$E(u,\mathcal{G}) := \frac{1}{2}||u'||^2_{L^2(\mathcal{G})} - \frac{1}{p}||u||^p_{L^p(\mathcal{G})}, \quad 2 < p \leq 6$$

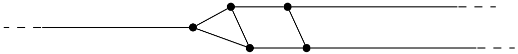
on a metric graph $\mathcal{G}$, is it possible to find a global minimizer (i.e. a ground state) among all the continuous functions that share the same mass μ, i.e.

$$||u||^2_{L^2(\mathcal{G})} = \mu?$$



Metric graphs: topology and metric.

- A metric graph $\mathcal{G}$ is a **connected** structure made of either finite or infinite **edges**, meeting at **vertices**. Each **bounded** edge $e \in \mathcal{E}$ can be identified with an interval $[0, \ell_e]$ and each **unbounded** one with an halfline $[0, +\infty)$.
- The metric structure is given by the **shortest-path** distance.



$\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Hence, $u \in L^p(\mathcal{G})$ is a L^p-function on every edge, while $u \in H^1(\mathcal{G})$ is a H^1-function on every edge and continuous in all vertices.

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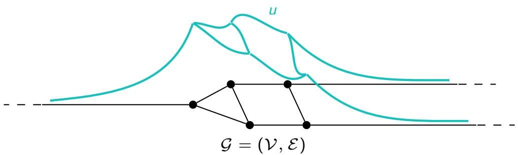

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About the functional, the constraint and the nonlinearity.

$$E(u,\mathcal{G}) = \frac{1}{2}||u'||^2_{L^2(\mathcal{G})} - \frac{1}{p}||u||^p_{L^p(\mathcal{G})}, \quad 2 < p \leq 6.$$

- E is not lower bounded! Fixed $u \in H^1(\mathcal{G})$, for $p > 2$ it follows

$$E(\lambda u,\mathcal{G}) = \frac{\lambda^2}{2}||u'||^2_{L^2(\mathcal{G})} - \frac{\lambda^p}{p}||u||^p_{L^p(\mathcal{G})} \to -\infty, \text{ for } \lambda \to +\infty.$$

- Fix $u \in H^1(\mathcal{G})$ with mass μ and consider $u_\lambda(x) = \sqrt{\lambda}u(\lambda x)$.

$$E(u_\lambda,\mathcal{G}) = \frac{\lambda^2}{2}||u'||^2_{L^2(\mathcal{G})} - \frac{\lambda^{\frac{p}{2}-1}}{p}||u||^p_{L^p(\mathcal{G})}.$$

  - $p \in (2,6)$ (subcritical case): E turns out to be lower bounded.
  - $p > 6$ (supercritical case): $E(u_\lambda,\mathcal{G}) \to -\infty$, as $\lambda \to +\infty$.
  - $p = 6$ (critical case): is E lower bounded? It depends on μ.

In conclusion, where does "Schrödinger" come from?

If u is a **ground state** for the constrained energy functional, it follows that:

$$\nabla E(u, \mathcal{G}) = \frac{\omega}{2} \nabla (\mu - ||u||^2_{L^2(\mathcal{G})}).$$

In particular, on $\mathcal{G}$ it holds the **stationary equation**

$$H\phi - |\phi|^{p-2}\phi + \omega\phi = 0,$$

associated to the **time-dependent nonlinear Schrödinger equation**

$$i\partial_t \psi = H\psi - |\psi|^{p-2}\psi,$$

where H is a self-adjoint extension of the Laplace operator.

Thanks for your attention!