Special geometric structures in six and seven dimensions

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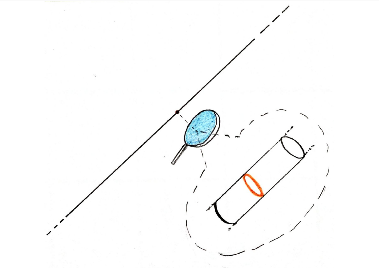
Motivations - String theory

- Phisical motivations: Unifying phisical theory

$$\text{String theory} \begin{cases} \text{General relativity} \\ \text{Quantum mechanics} \end{cases}$$

Heterotic string theory: 10 dimensions

Ansatz [Hull-Strominger] $T^{9,1} = \mathbb{R}^{3,1} \times M^6$



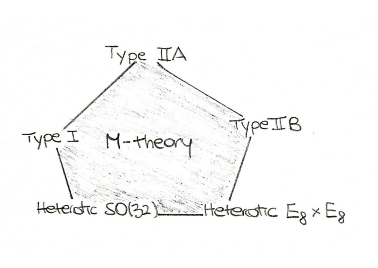
"If we were to take a detailed look at our four-dimensional spacetime, as depicted by the line in this figure, we'd see it's actually harboring six extra dimensions, curled up in an intricate though minuscule geometric space known as a Calabi-Yau manifold. [...] No matter where you slice this line, you will find a hidden Calabi-Yau, and all the Calabi-Yau manifolds exposed in this fashion would be identical."
–Shing Tung Yau

Goal: understanding possibile concrete manifestations of this model (constraints on M^6) $\rightsquigarrow$ Strominger system
Special solutions: Parallel SU(3)-structures

Motivations - String theory

$$\text{5 string theories:} \begin{cases} \text{Type IIA} \\ \text{Type IIB} \\ \text{Type I} \\ \text{Heterotic } E_8 \times E_8 \\ \text{Heterotic } SO(32) \end{cases} \qquad (10\,\text{dimensions} \rightsquigarrow 6\,\text{extra})$$

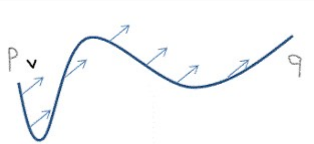
$\Longrightarrow$ **M-theory** [Witten, 1995]: all theories are tied together through a common framework (11 dimensions $\rightsquigarrow$ 7 extra) $\rightsquigarrow$ special solutions: G_2 manifolds



"So one might think that much of what we've talked about so far [...] could have been suddenly rendered obsolete by Witten's eureka moment. Fortunately, [...] that is not the case. [...] First, eleven-dimensional spacetime is treated as the product of ten-dimensional spacetime and a one-dimensional circle. We compactify the circle, [...] then take those ten dimensions and compactify on a Calabi-Yau manifold, as usual, to get down to the four dimensions of our world."
–Shing Tung Yau

"So even in M-theory, Calabi-Yau manifolds are still in the center of things."
–Petr Horava

M n-fold, $p, q \in M$, $v \in T_pM$
$\gamma : [0,1] \to M$ curve, $\gamma(0) = p$, $\gamma(1) = q$
Question: Is there a way to move v parallel along γ?



Simple case $M = \mathbb{R}^n$
$\rightsquigarrow$ choose a vector field $v(t)$ along $\gamma(t)$ starting from v such that
$$v'(t_0) := \lim_{t \to t_0} \frac{v(t) - v(t_0)}{t - t_0} = 0, \quad \forall t_0 \in [0,1].$$

More generally Problem: $v(t) \in T_{\gamma(t)}M \neq T_pM$

We need a connection ∇, X vector field along γ;
X is parallel with respect to ∇ if $\nabla_{\frac{d\gamma}{dt}} X = 0$, $\frac{d}{dt}$ vector field on $[0,1]$.
Fact: $v \in T_pM \to \exists!\, X$ parallel vector field along γ, $X_p = v$

Parallel transport

$P_\gamma : T_pM \to T_qM$, $v \mapsto P_\gamma(v) :=$ endpoint of the unique parallel vector field along γ starting from v

Motivations - The Holonomy group of a Riemannian manifold

Holonomy group

$\mathrm{Hol}_p(\nabla) := \{P_\gamma \,|\, \gamma \text{ piecewise } C^\infty,\ \gamma(0) = \gamma(1) = p\}$

Fact: When M is connected, $\mathrm{Hol}_p(\nabla)$ does not depend on the base point and we can refer to it as $\mathrm{Hol}(\nabla) \subseteq \mathrm{GL}(\mathbb{R}, n)$, up to conjugation

(M, g) Riemannian manifold, $\nabla^{\mathrm{LC}} := \nabla$ Levi-Civita connection
$\implies \mathrm{Hol}(\nabla) \subseteq \mathrm{SO}(n)$ when M is simply connected

Theorem [Berger 1955]

Let (M, g) be a complete, simply connected, irreducible, non-symmetric Riemannian manifold of dimension n. Then $\mathrm{Hol}(\nabla)$ is one of the following groups:

- $\mathrm{SO}(n)$;
- $\mathrm{U}(m)$, with $n = 2m \geqslant 4$ (Kähler);
- $\mathrm{SU}(m)$, with $n = 2m \geqslant 4$ (Calabi-Yau, $\mathrm{Ric}(g) = 0$);
- $\mathrm{Sp}(m)\mathrm{Sp}(1)$, with $n = 4m \geqslant 8$ (hyperkähler, $\mathrm{Ric}(g) = 0$);
- $\mathrm{Sp}(m)$, with $n = 4m \geqslant 8$ (quaternionic Kähler, $\mathrm{Ric}(g) = c\,g$, $c \neq 0$);
- G_2, with $n = 7$ (exceptional holonomy G_2, $\mathrm{Ric}(g) = 0$);
- $\mathrm{Spin}(7)$, with $n = 8$ (exceptional holonomy $\mathrm{Spin}(7)$, $\mathrm{Ric}(g) = 0$).

Definitions - SU(3)-structures

$(\mathbb{R}^6)^* = \langle e^1, \dots, e^6 \rangle$
$\omega_0 = e^{12} + e^{34} + e^{56}$
$\rho_0 = e^{135} - e^{146} - e^{236} - e^{245}$

The Lie group SU(3)

$\mathrm{SU}(3) := \{f \in \mathrm{GL}(6, \mathbb{R}) \mid f^*\omega_0 = \omega_0, f^*\rho_0 = \rho_0\} \subset \mathrm{SO}(6)$

SU(3) is a compact, connected, simply connected, simple Lie group with $\dim_{\mathbb{R}} \mathrm{SU}(3) = 8$

SU(3)-structures

M 6-fold, $(\omega, \rho) \in \Lambda^2(M) \times \Lambda^3(M)$ is called a SU(3)-structure if

$$(T_pM, \omega_p, \rho_p) \cong (\mathbb{R}^6, \omega_0, \rho_0)$$

for any $p \in M$.

$$(\omega, \rho) \Longleftrightarrow (g, J, \mathrm{Vol}_g), \quad \text{where} \begin{cases} g \text{ Riemannian metric} \\ J \text{ almost complex structure} \\ \mathrm{Vol}_g \text{ orientation} \end{cases} \quad \text{on M}$$

Moreover, $g, \mathrm{Vol}_g \rightsquigarrow \nabla^{\mathrm{LC}}, *_g$ Hodge operator

Definitions - G_2-structures

$(\mathbb{R}^7)^* = \langle e^1, \dots, e^7 \rangle$
$\varphi_0 = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245}$

The Lie group G_2

$G_2 := \{ f \in GL(7, \mathbb{R}) \mid f^* \varphi_0 = \varphi_0 \} \subset SO(7)$

G_2 is a compact, connected, simply connected, simple Lie group with $\dim_{\mathbb{R}} G_2 = 14$

G_2-structures

N 7-fold, $\varphi \in \Lambda^3(M)$ is called a G_2-structure if

$$(T_p N, \varphi_p) \cong (\mathbb{R}^7, \varphi_0)$$

for any $p \in N$.

φ induces a Riemannian metric g_φ and an orientation Vol_{g_φ} on N:

$$g_\varphi(X, Y)\mathrm{Vol}_{g_\varphi} = \frac{1}{6} \iota_X \varphi \wedge \iota_Y \varphi \wedge \varphi.$$

Moreover, $g_\varphi, \mathrm{Vol}_{g_\varphi} \rightsquigarrow \nabla^{\mathrm{LC}}, *_{g_\varphi}$ Hodge operator

Link between SU(3) and G_2-structures

$$\mathbb{R}^7 = \mathbb{R}^6 \times \mathbb{R}$$
$$\omega_0 = e^{12} + e^{34} + e^{56}$$
$$\rho_0 = e^{135} - e^{146} - e^{236} - e^{245}$$

$$\begin{aligned} \implies \varphi_0 &= \omega_0 \wedge e^7 + \rho_0 \\ &= e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245} \end{aligned}$$

A G_2-structure φ on a 7-fold induces an SU(3)-structure (ω, ρ) on every oriented hypersurface and, vice versa, an SU(3)-structure (ω, ρ) on a 6-manifold M induces a G_2-structure φ on the cartesian product $M \times L$, $L = \mathbb{R}, S^1$.

M 6-fold

Let (ω, ρ) be an SU(3)-structure on M and let d be the De Rham differential of M

Closed SU(3)-structures

$d\rho = 0$

Coclosed SU(3)-structures

$d *_g \rho = 0$

Symplectic SU(3)-structures

$d\omega = 0$

Torsion free SU(3)-structures

$$\begin{cases} d\rho = 0 \\ d\omega = 0 \\ d *_g \rho = 0 \end{cases} \iff \mathrm{Hol}(\nabla^{\mathrm{LC}}) \subseteq \mathrm{SU}(3) \implies \mathrm{Ric}(g) = 0$$

Special G_2-structures

N 7-fold

Let φ be a G_2-structure on N and denote by d the De Rham differential of N

Closed G_2-structure

$d\varphi = 0$

Coclosed G_2-structure

$d *_\varphi \varphi = 0$

Torsion free G_2-structure

$$\begin{cases} d\varphi = 0 \\ d *_\varphi \varphi = 0 \end{cases} \iff \mathrm{Hol}(\nabla^{\mathrm{LC}}) \subseteq \mathrm{G}_2 \implies \mathrm{Ric}(g_\varphi) = 0$$

Closed G_2-structures

Closed G_2-structures are a potential tool to obtain new examples of torsion free G_2-structures on compact manifolds

- [Joyce 1996]: On a compact 7-fold a closed G_2-structure with small torsion can be deformed into a torsion free one;
- [Bryant 2006]: Laplacian flow for closed G_2-structures.

$\rightsquigarrow$ **CLASSIFICATION PROBLEMs:** a central problem in differential geometry!

- Closed G_2-structures on special classes of 7-folds (Lie groups, Homogeneous spaces)
- Special solutions of the Laplacian flow (self-similar solutions)

$\rightsquigarrow$ **Some other problems:**

- Special classes of non-integrable SU(3)-structures on 6-folds (cohomogeneity one manifolds)
- Properties of the Hitchin flow for half-flat SU(3)-structures

Further readings

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